Lower Bounds for Subgraph Detection in the CONGEST Model

TZLIL GONEN

JOINT WORK WITH ROTEM OSHMAN

Tel Aviv University

Introduction: Subgraph Freeness

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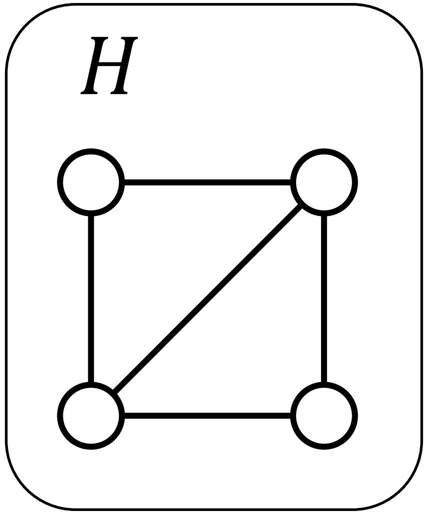

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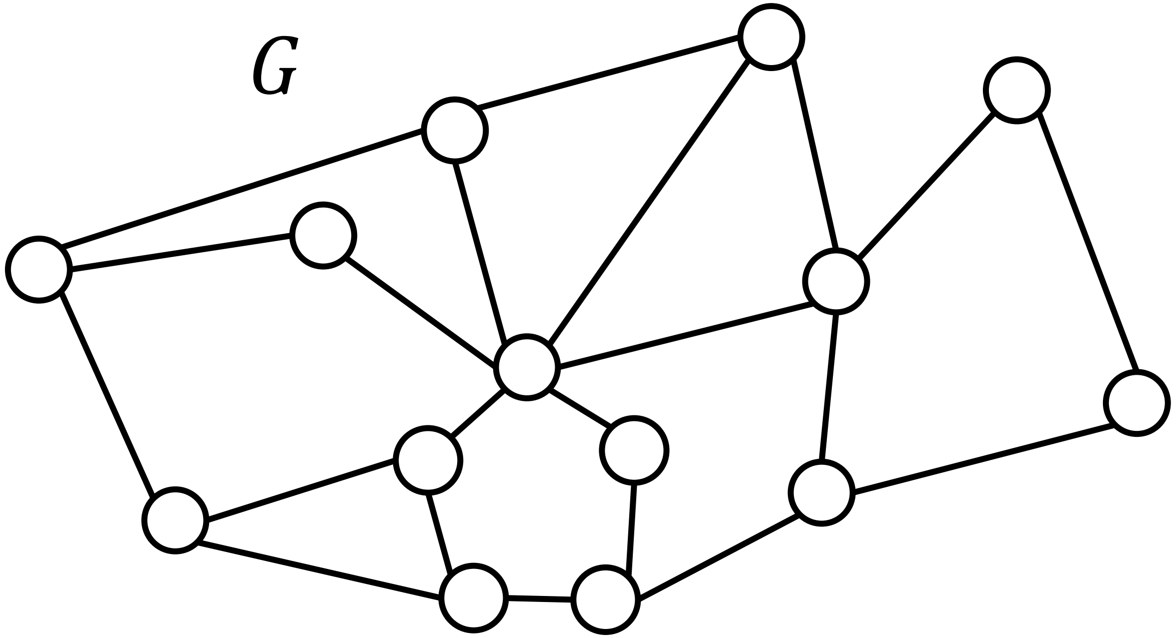

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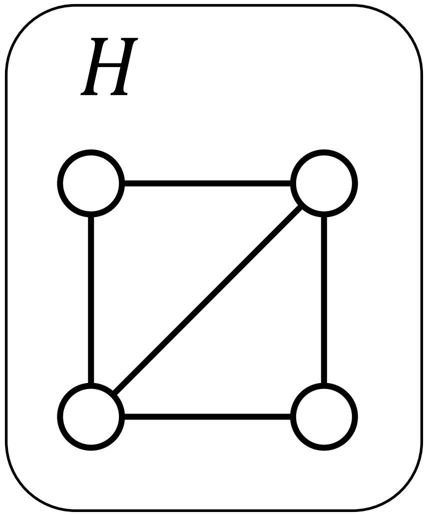

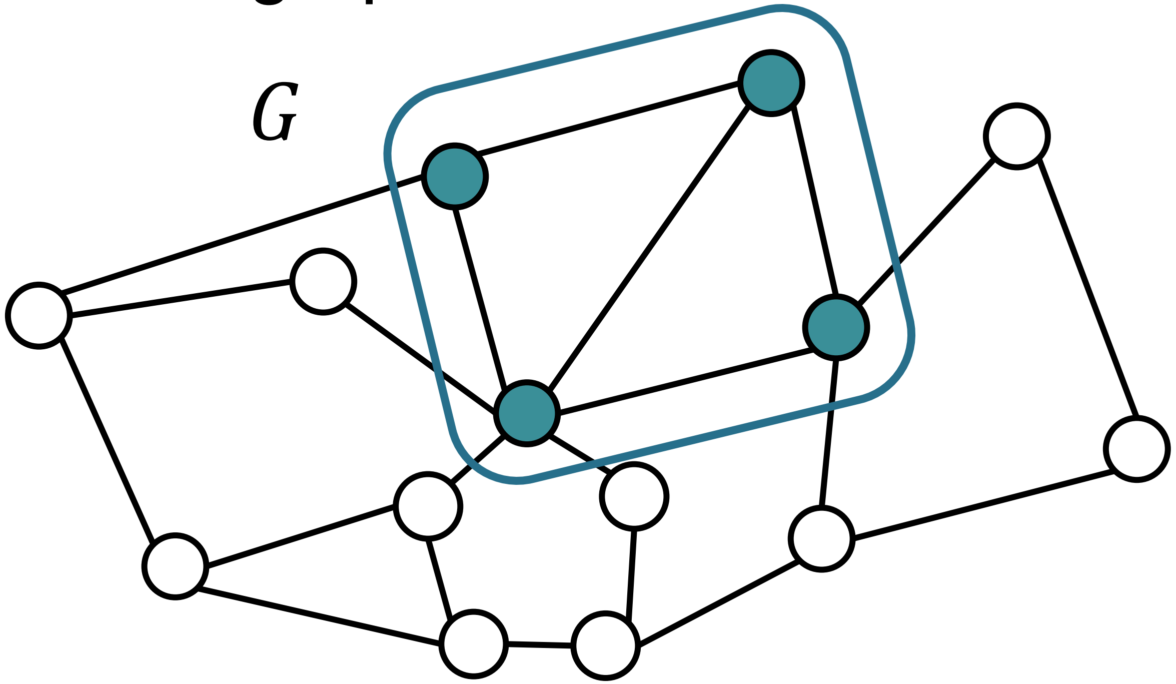

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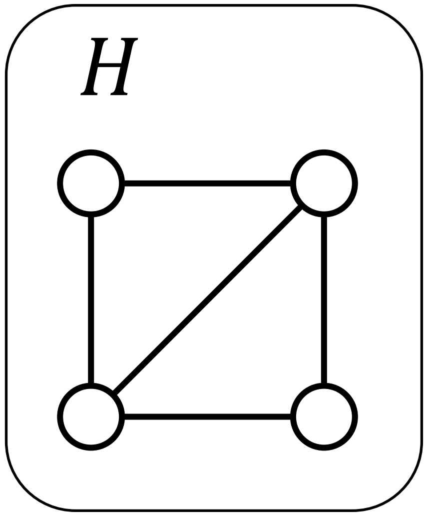

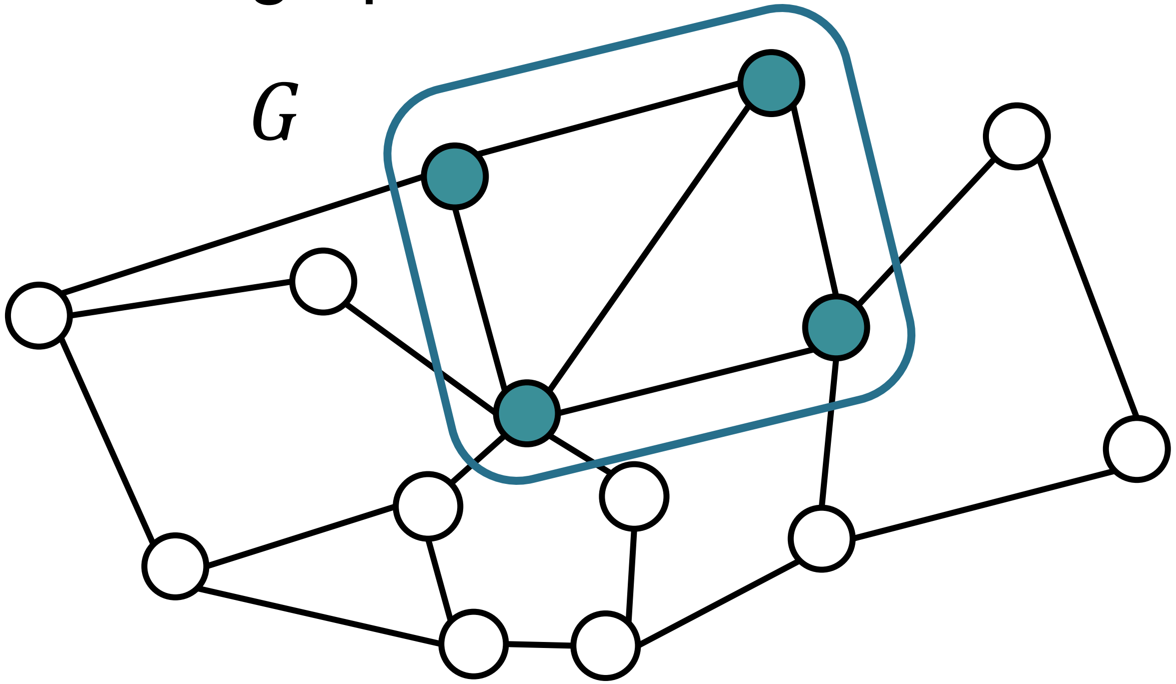

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- Our question: what else?

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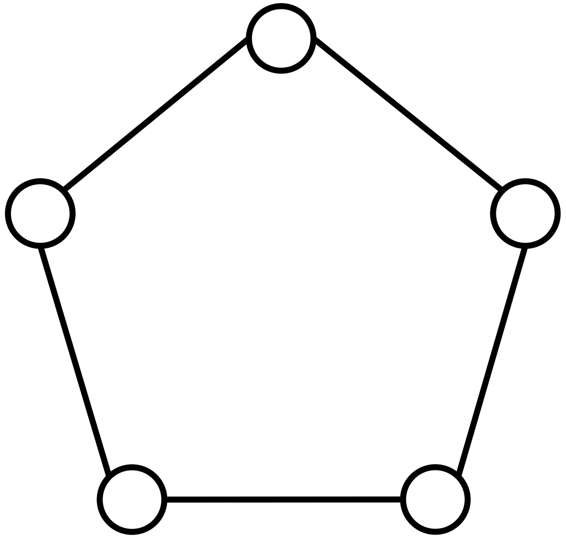

$\tilde{\Omega}(n)$ rounds
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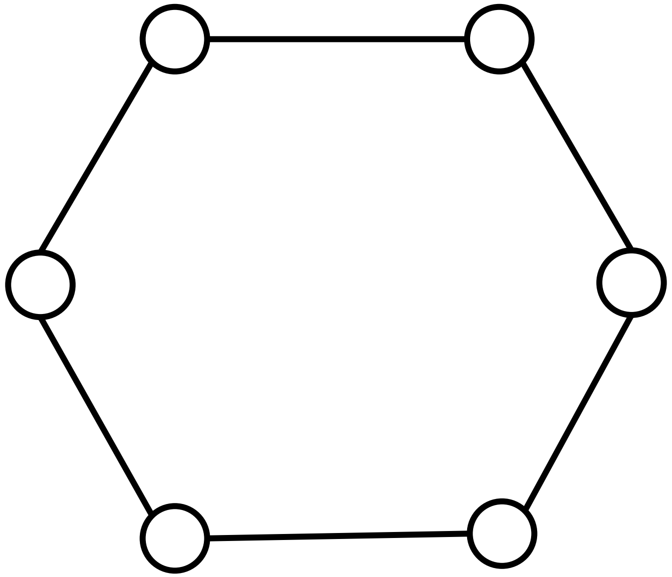

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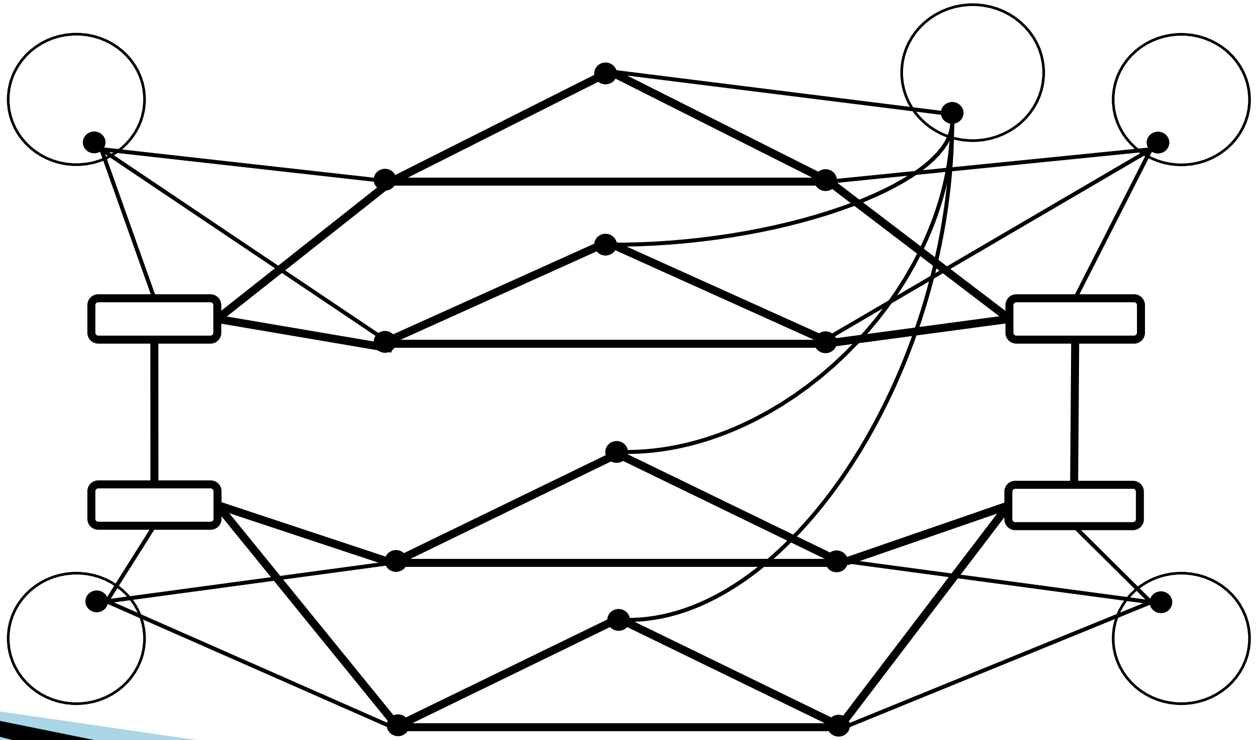

$\tilde{\Omega}\left(n^{2-1/k}\right)$ rounds for size $\Theta(k)$ graph

[Fischer, Gonen, Oshman '17]

Reduction #1: Vertex Replacement

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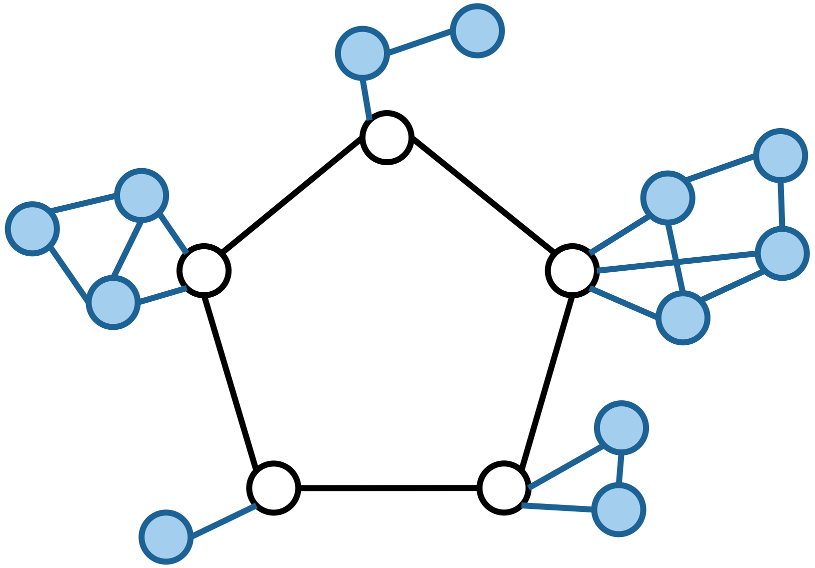
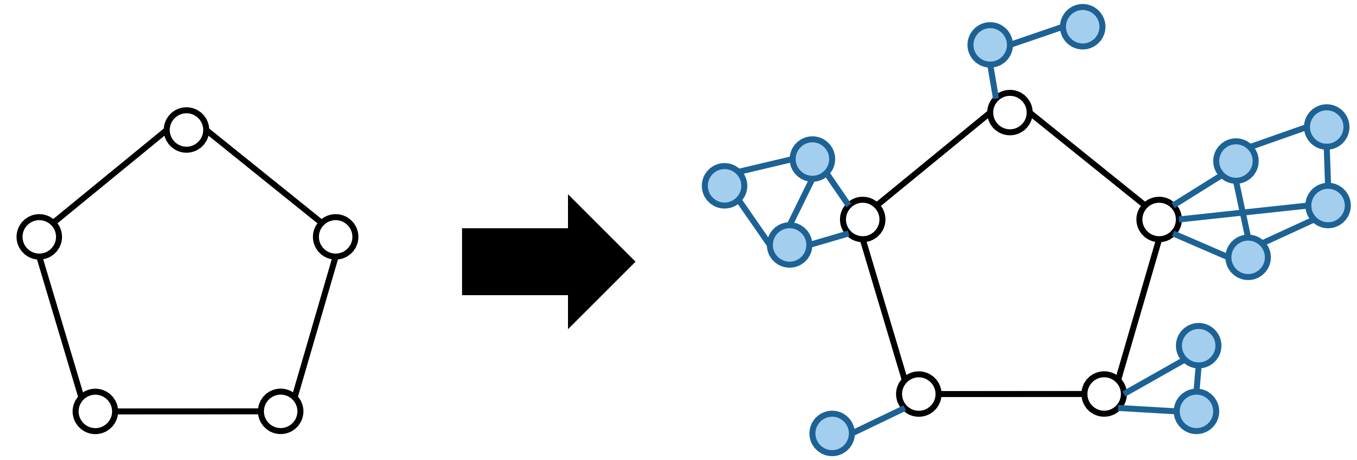


Reduction #1: Vertex Replacement

- Take any 2-connected graph for which a lower bound is known…
- Attach arbitrary connected graph to each vertex…
- Obtain nearly the same lower bound



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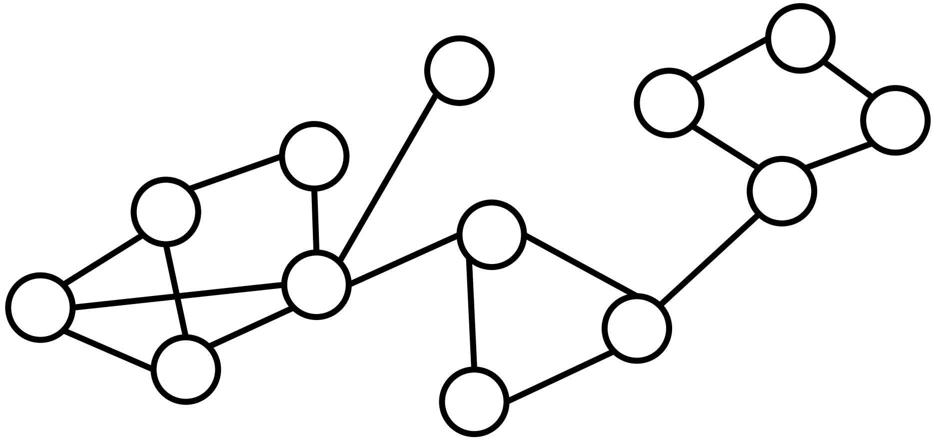

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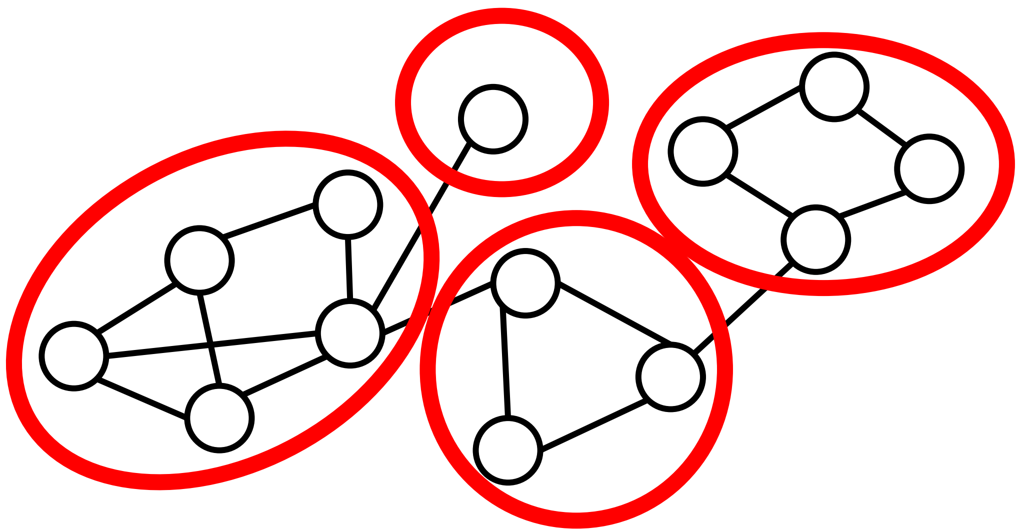
- Consider any connected graph H
- Decompose H into a tree of 2-connected components $C_1, \dots, C_k$



Corollary from Reduction #1

- Consider any connected graph H
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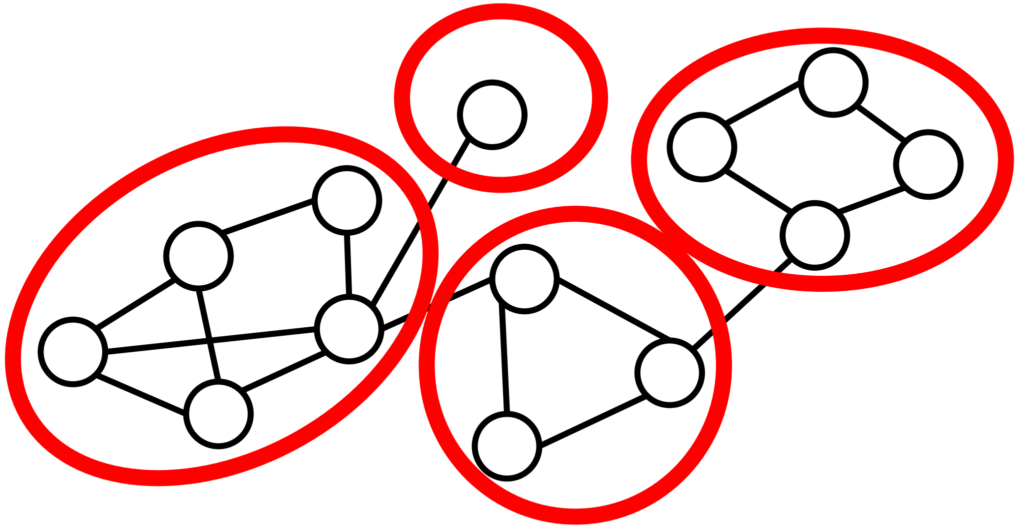
- Hardness of H-freeness $\geq \max_i$ (hardness of C_i-freeness)



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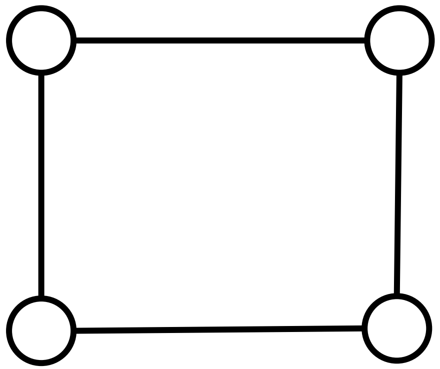
- Replace each edge by an arbitrary graph s.t. the result is 2-connected



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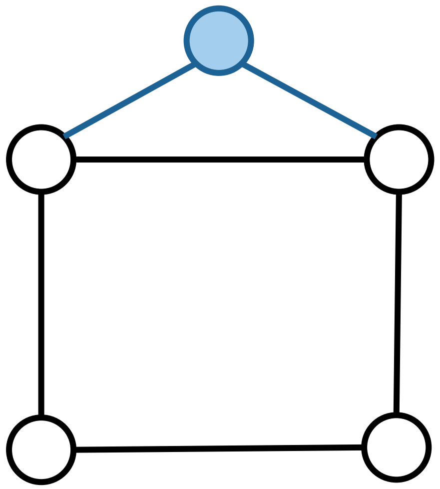

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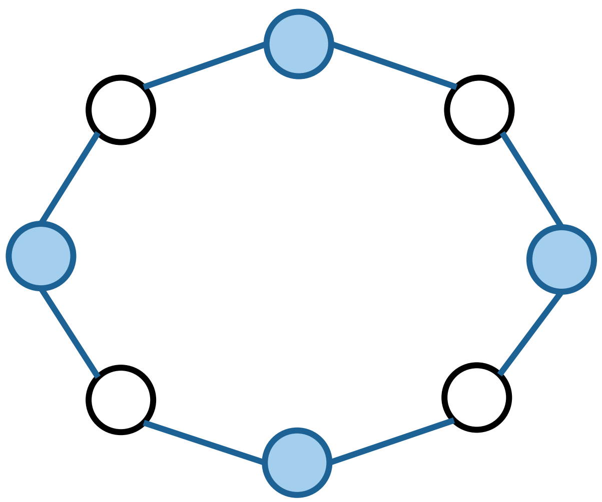

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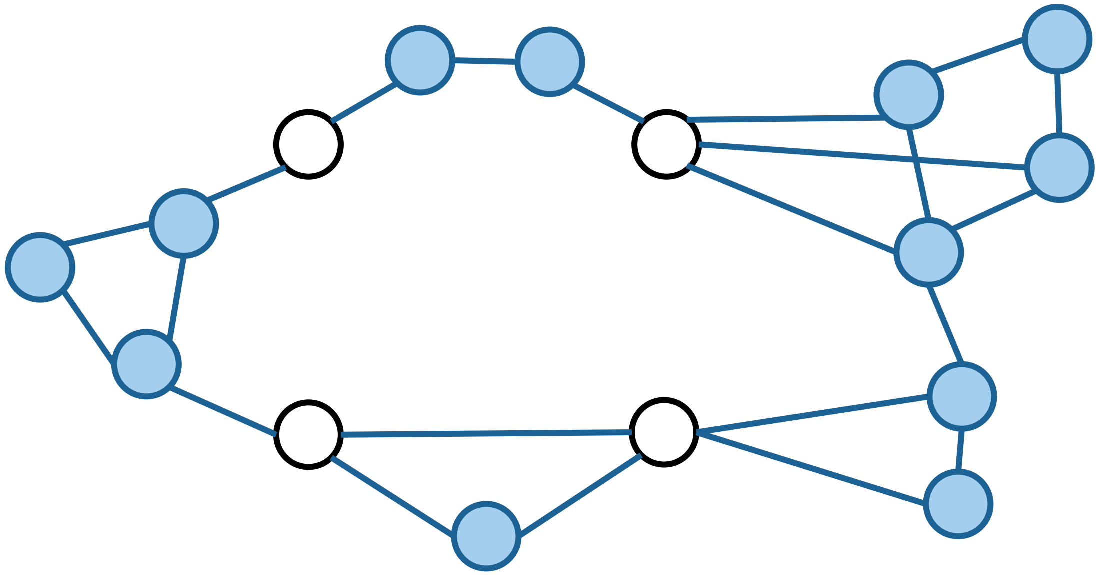

Reduction #2: Edge Replacement

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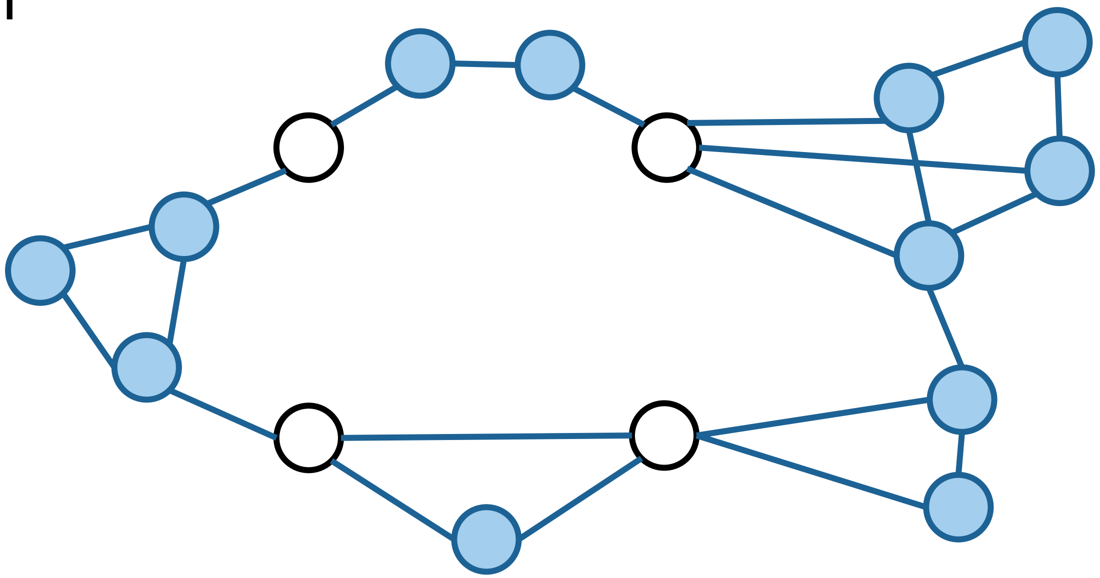
- Obtain lower bound of $\Omega(n^{\delta})$ rounds
- $\delta \in (0, 1/2]$ depends on $|H|$



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Vertex Replacement Reduction

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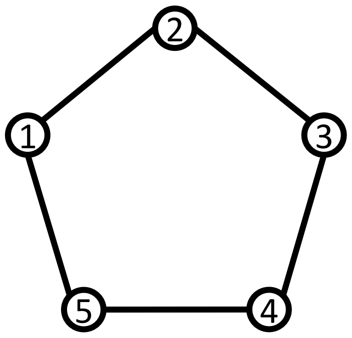
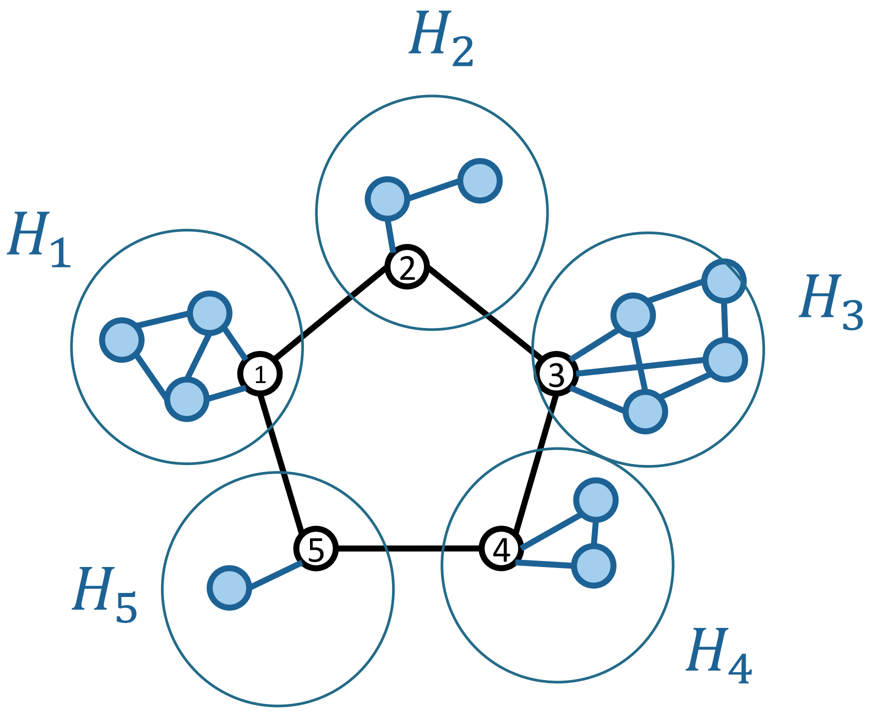
- Notation:



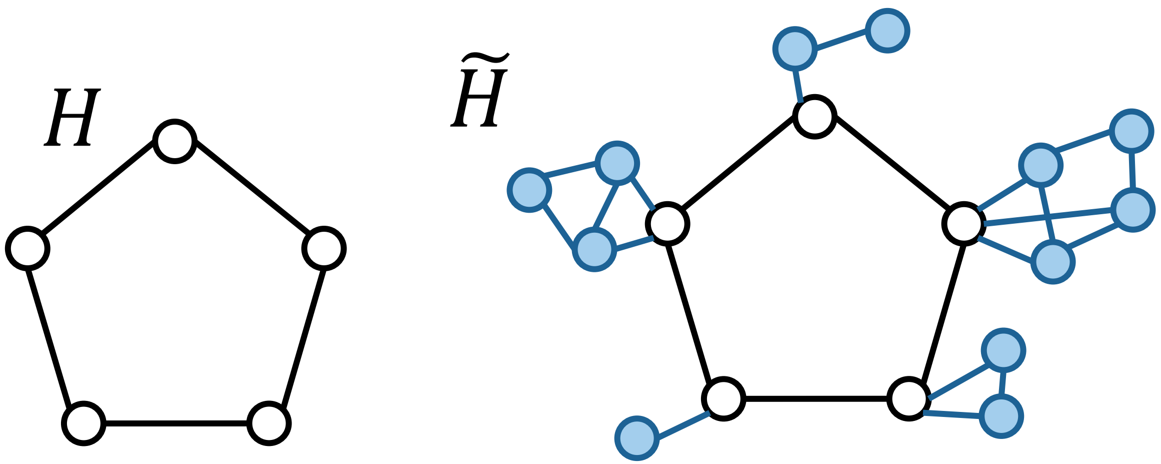


H: Graph we know a lower bound for

$\widetilde{H}$: Graph we want a lower bound for

Vertex Replacement Reduction

- Fix an algorithm $\tilde{A}$ for testing $\widetilde{H}$-freeness



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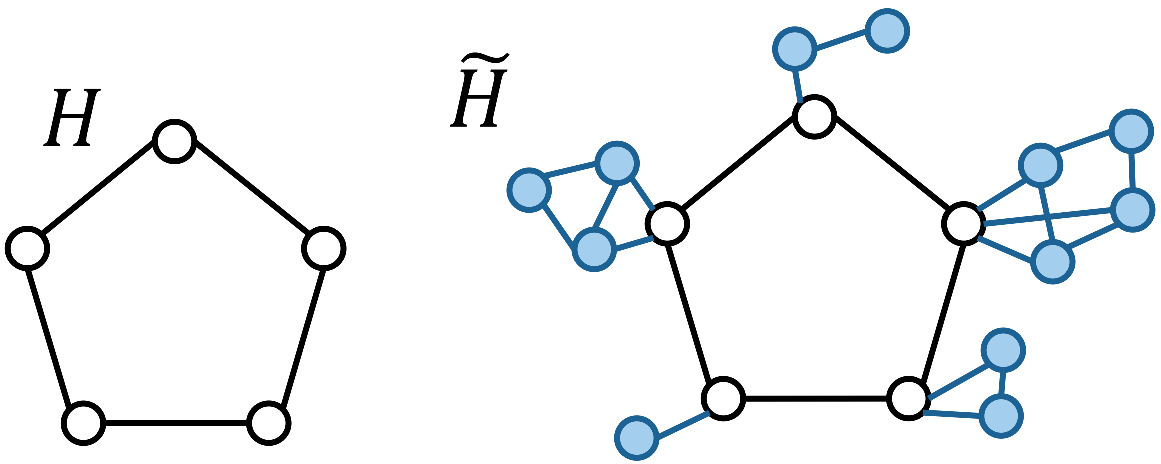

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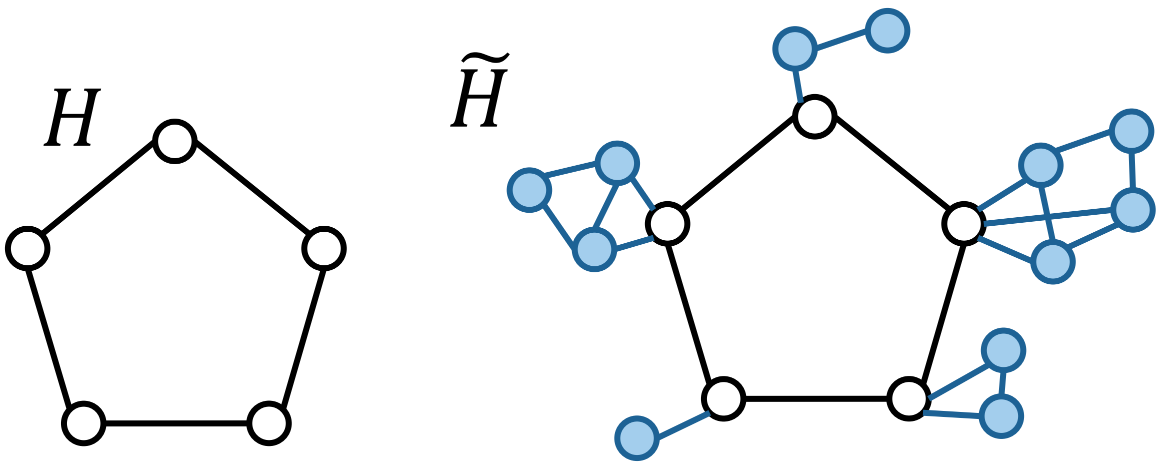

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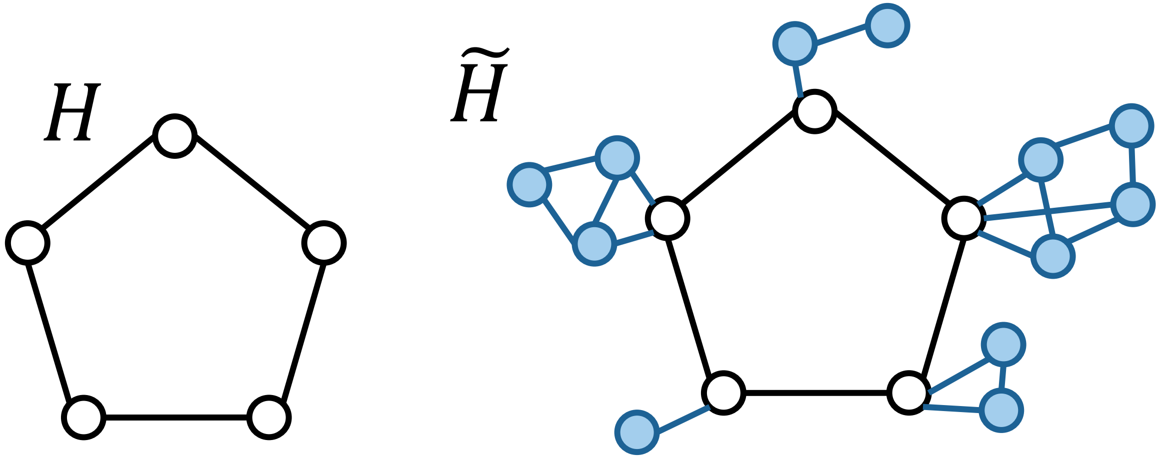

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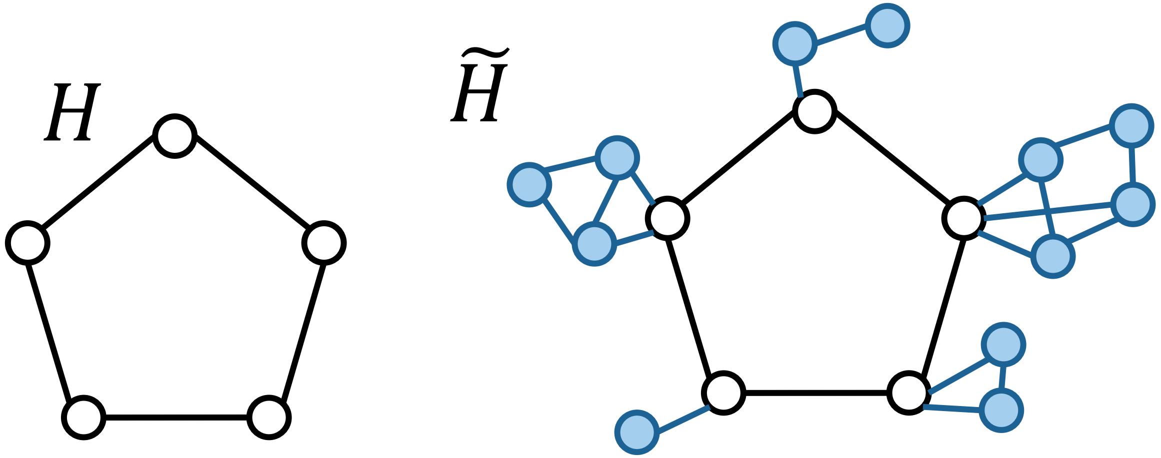

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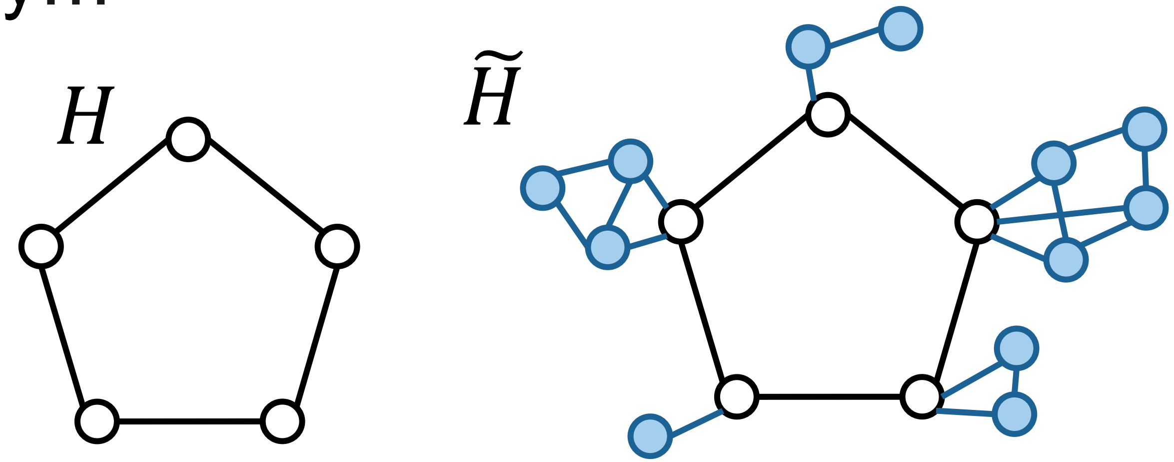
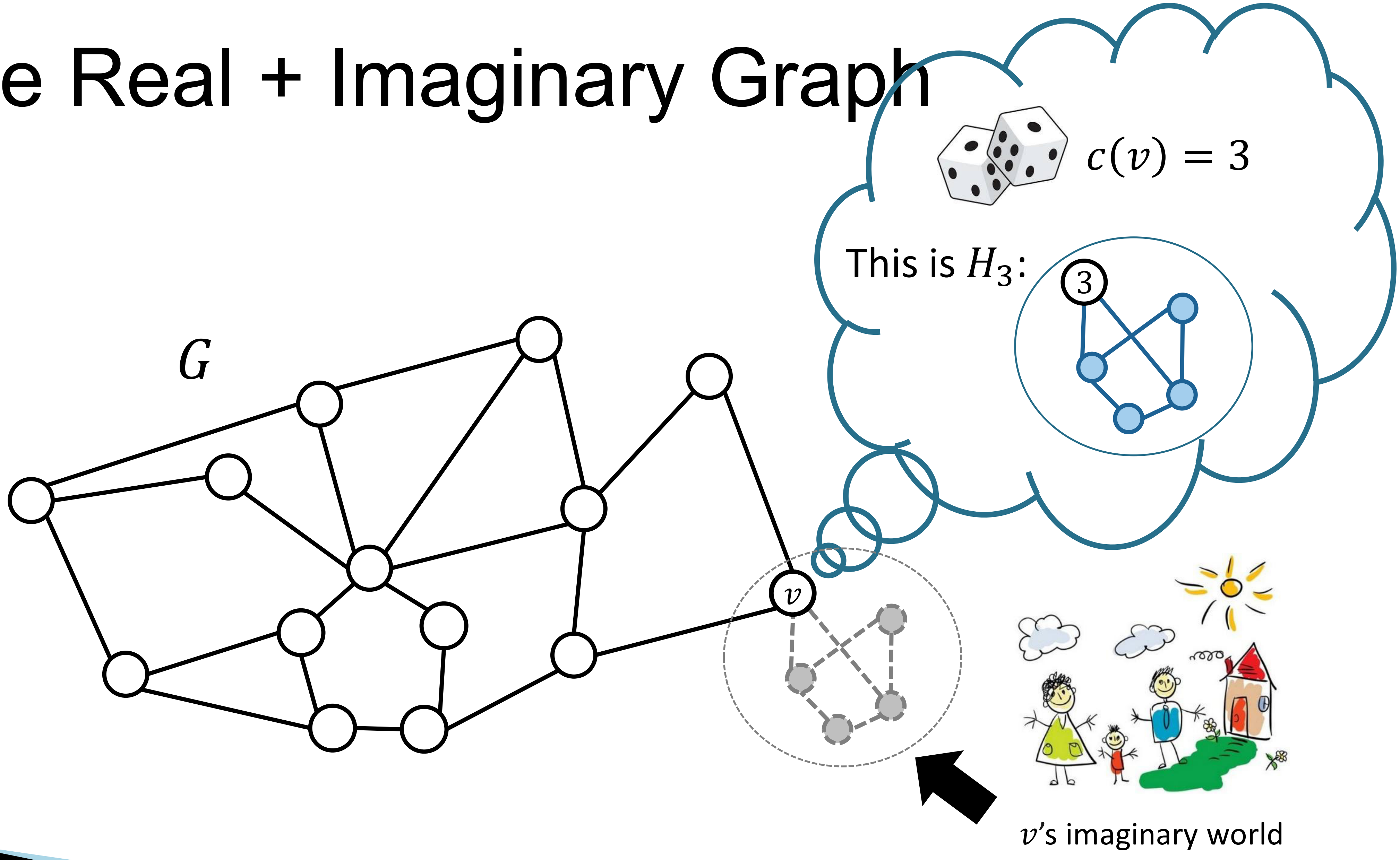
- How to choose a role? Randomly…



The Real + Imaginary Graph



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- Reduction can be derandomized
  - Additional $O(\log n)$ factor

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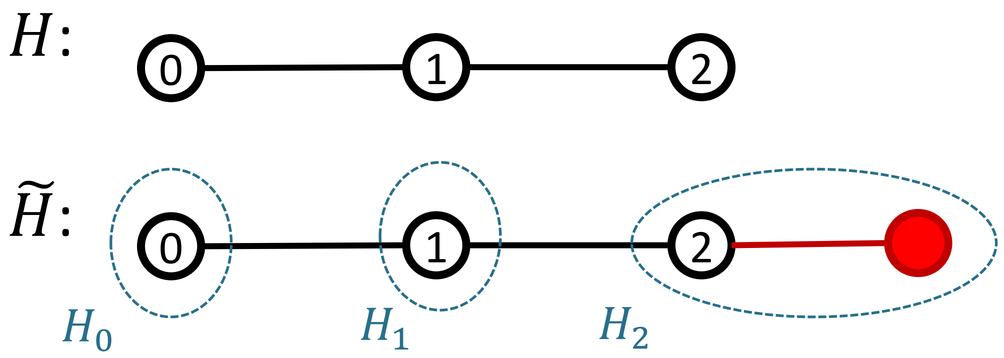

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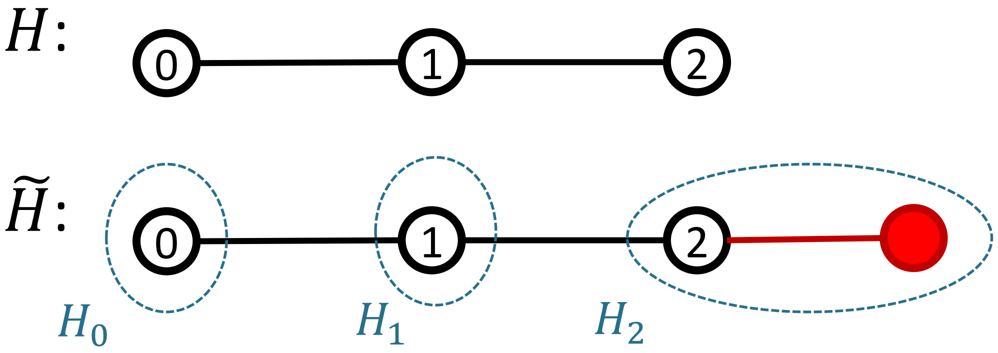

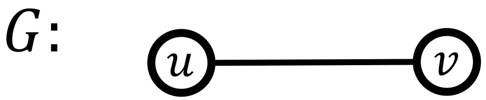

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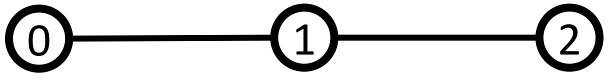
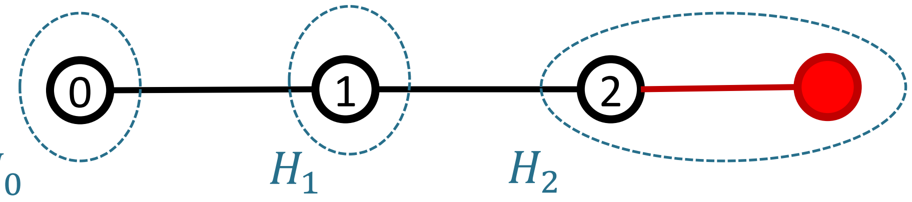


$\widetilde{H}$:



G:

G does not contain H! But…

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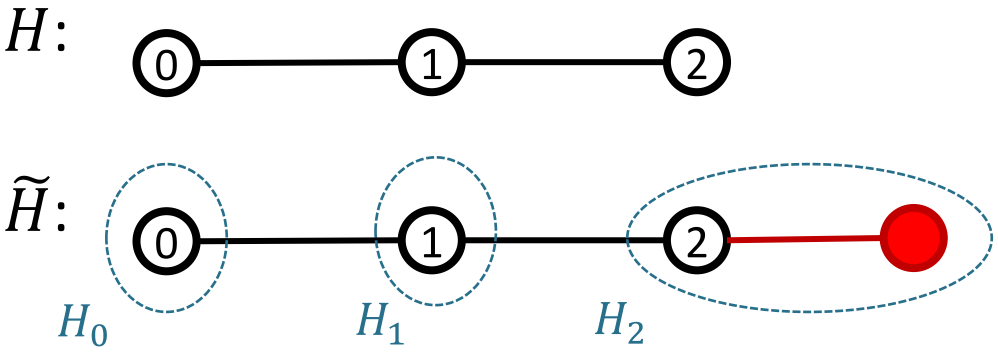

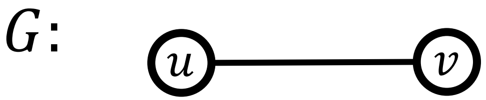

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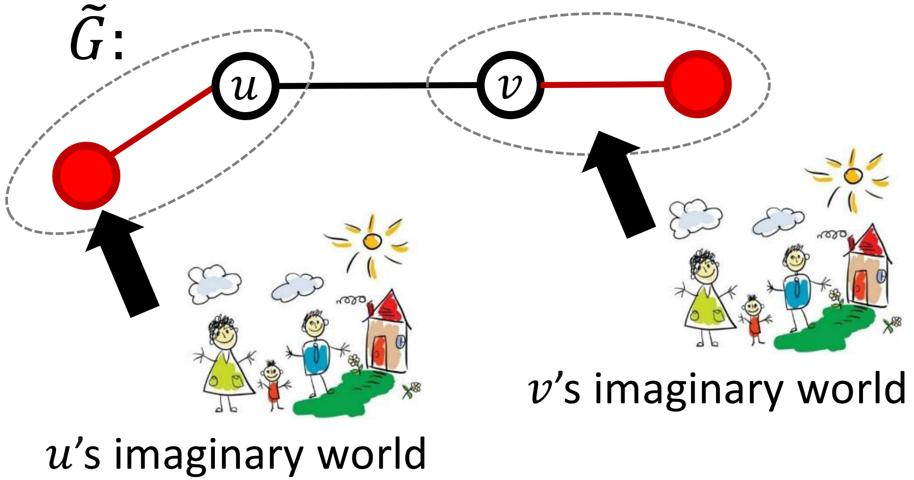

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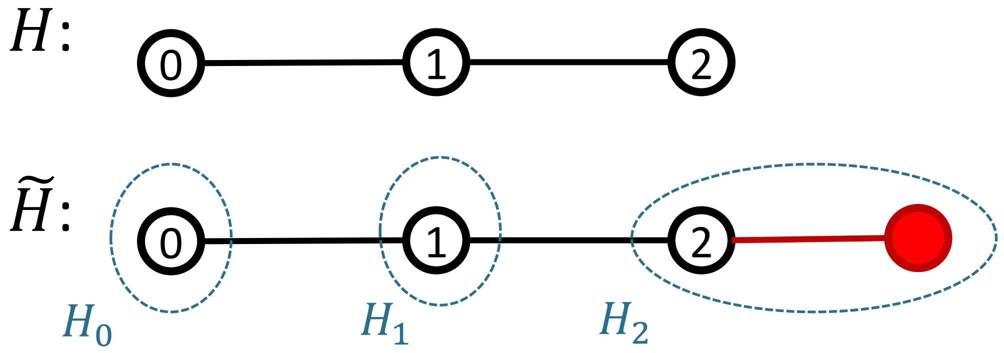

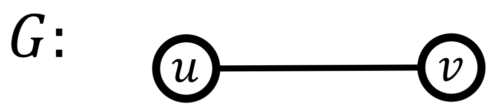

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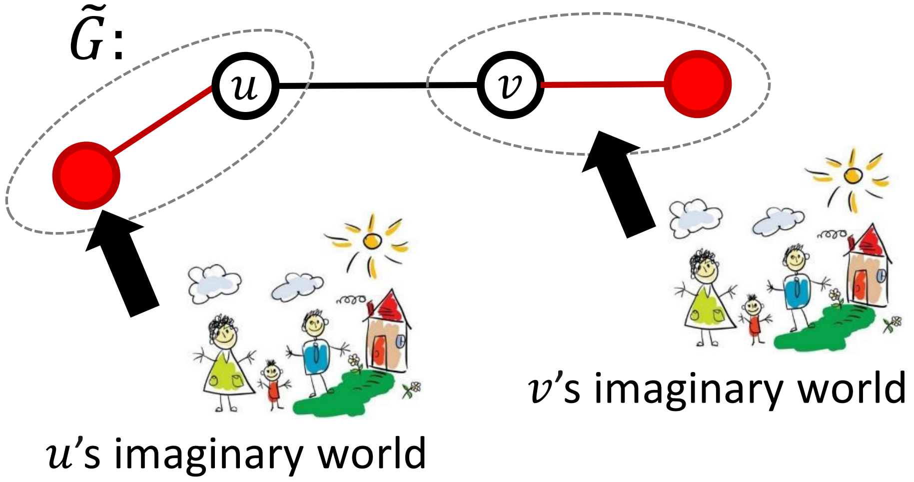

$\tilde{G}$ contains $\widetilde{H}$…

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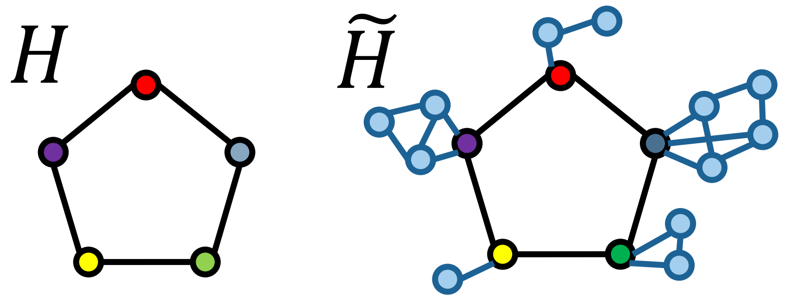
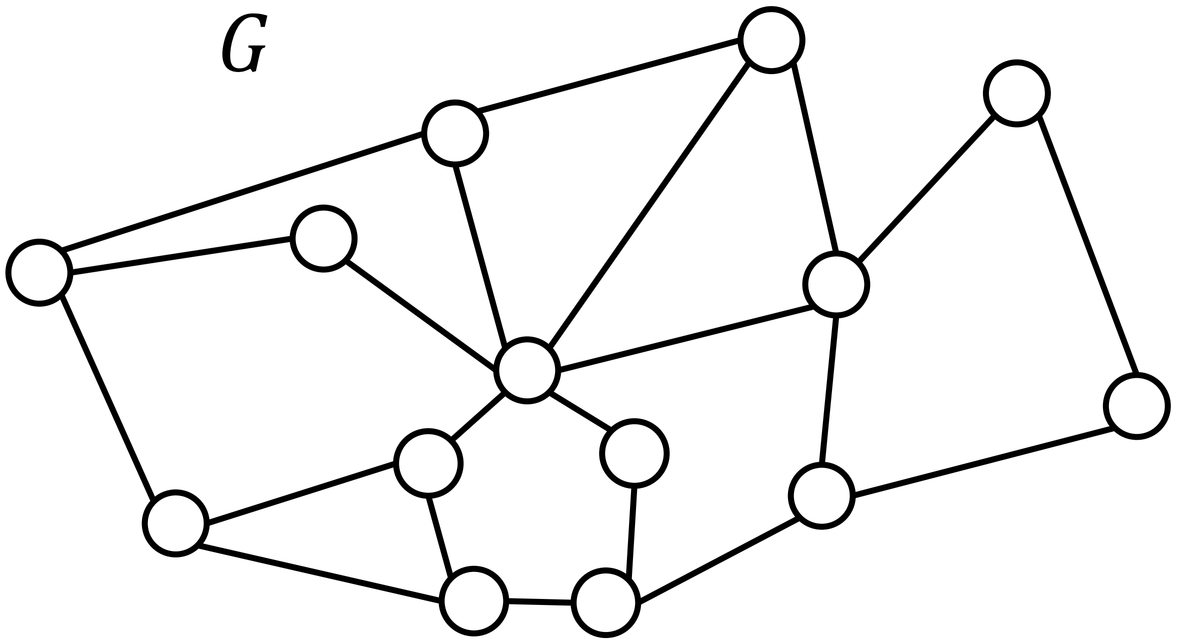


- Suppose G contains H as a subgraph



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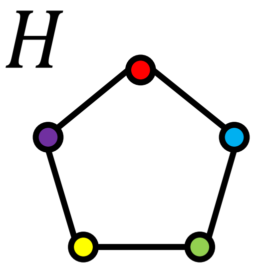
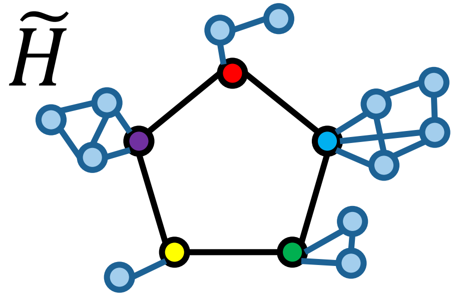
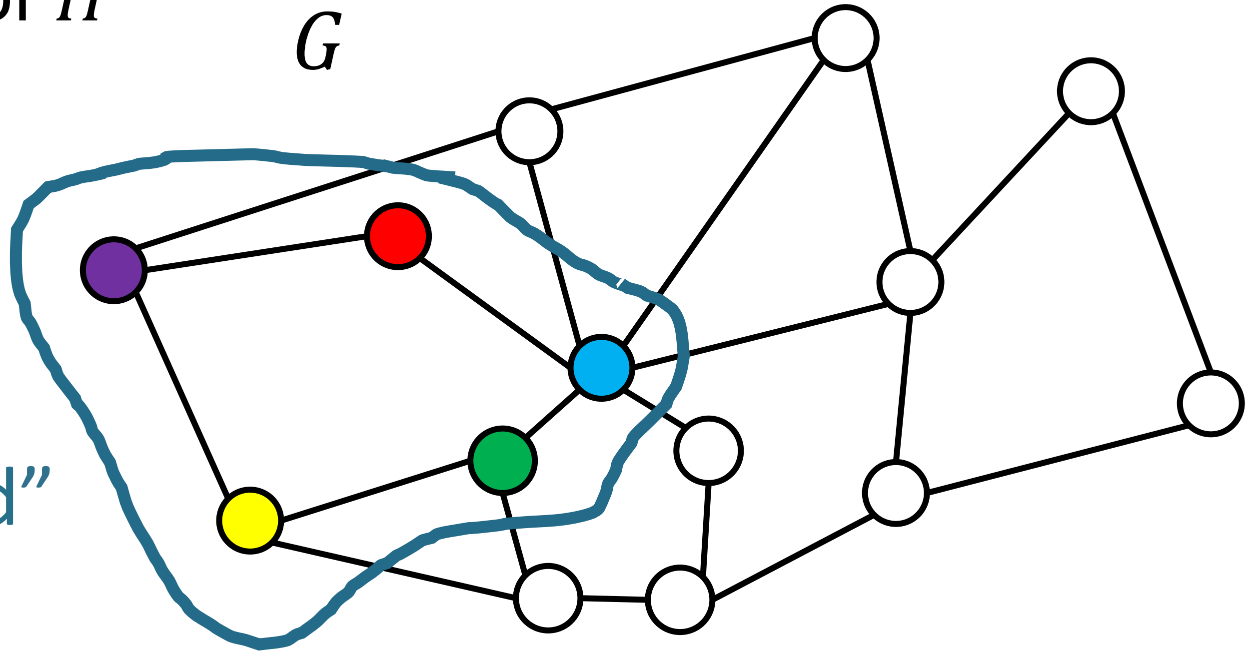




- Suppose G contains H as a subgraph
- W.p. $\geq 1/k^k$, G contains a properly colored copy of H



"Properly colored"

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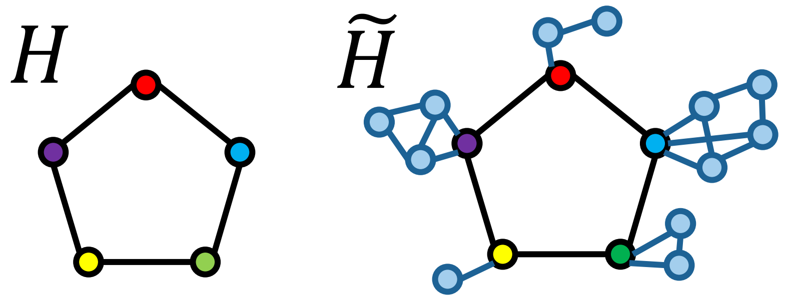

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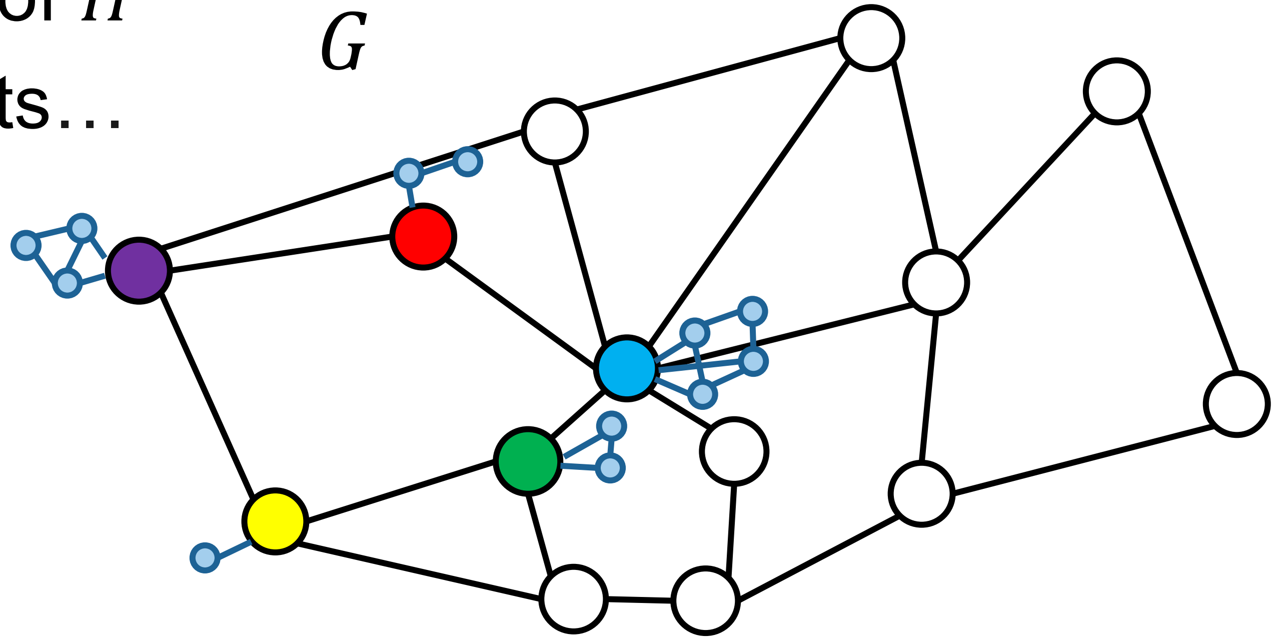
- After adding imaginary parts…
- A copy of $\widetilde{H}$ appears



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  - Let $\sigma(\widetilde{H})$ be the image of $\widetilde{H}$ inside $\tilde{G}$

Correctness Proof

- Now suppose G does not contain H
- We show that for any coloring of G, resulting $\tilde{G}$ does not contain $\widetilde{H}$
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  - Let $\sigma(H)$ be the image of H inside $\sigma\left(\widetilde{H}\right)$
    - G is H-free $\Rightarrow \sigma(H)$ must contain some imaginary nodes!

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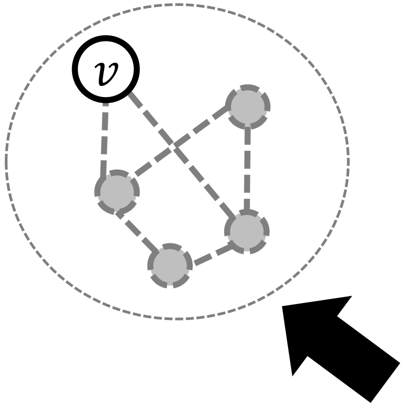
  - Let $\sigma(H)$ be the image of H inside $\sigma(\widetilde{H})$
    - G is H-free $\Rightarrow \sigma(H)$ must contain some imaginary nodes!
- Proof strategy: show that $\tilde{G}$ contains infinitely many copies of H…

Correctness Proof

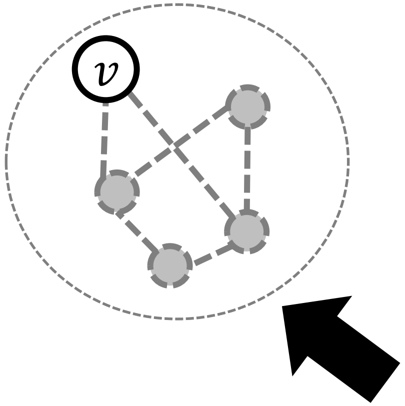
- Lemma 1: $\sigma(H)$ is inside the imaginary world of some vertex v



v's imaginary world

Correctness Proof

- Lemma 1: $\sigma(H)$ is inside the imaginary world of some vertex v
- Proof:
  - We said: $\sigma(H)$ intersects some vertex v's imaginary world
  - Can't reach rest of the world except through v



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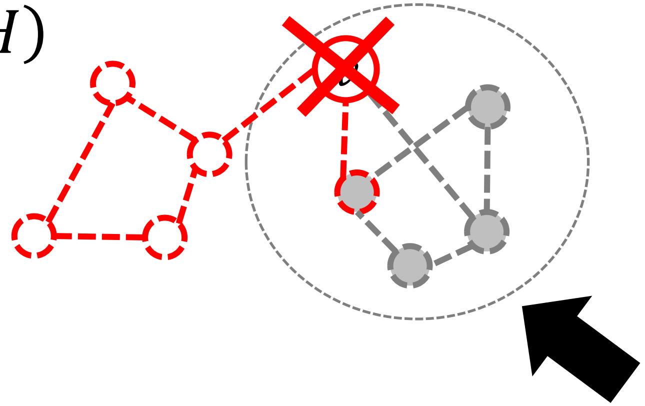
  - Removing v disconnects $\sigma(H)$



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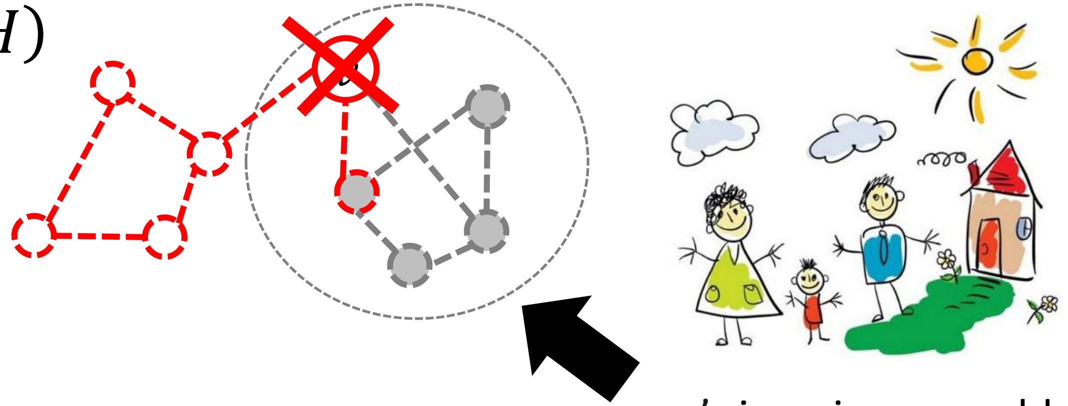
  - … but H is 2-connected

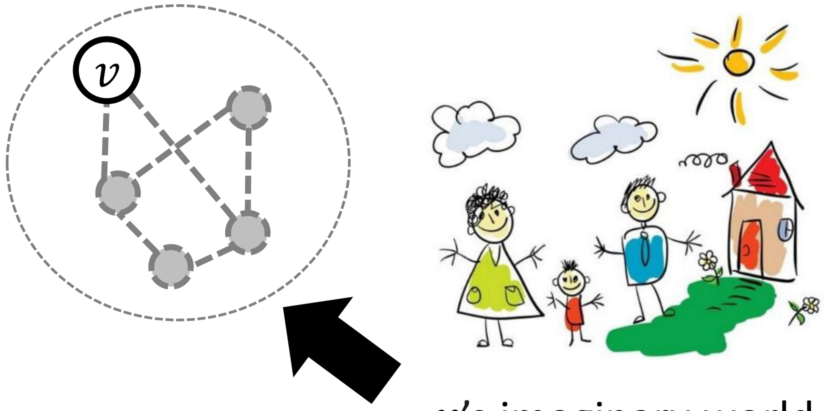


v's imaginary world

Correctness Proof

✓<u>Lemma 1:</u> $\sigma(H)$ is inside the imaginary world of some vertex v

- <u>Lemma 2:</u> $\sigma\left(H_{c(v)}\right) \not\subseteq v$'s imaginary world
- <u>Proof:</u> not enough room in there…



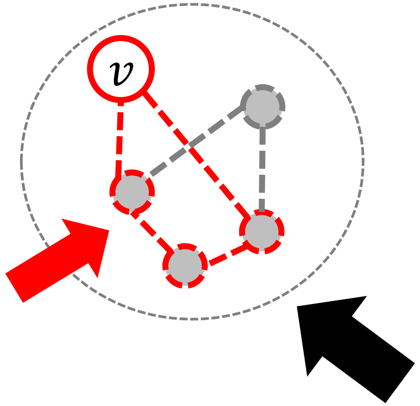
v's imaginary world

Correctness Proof

✓<u>Lemma 1:</u> $\sigma(H)$ is inside the imaginary world of some vertex v

- <u>Lemma 2:</u> $\sigma\left(H_{c(v)}\right) \nsubseteq v$'s imaginary world
- <u>Proof:</u> not enough room in there…

imaginary world $\sim H_{c(v)}$

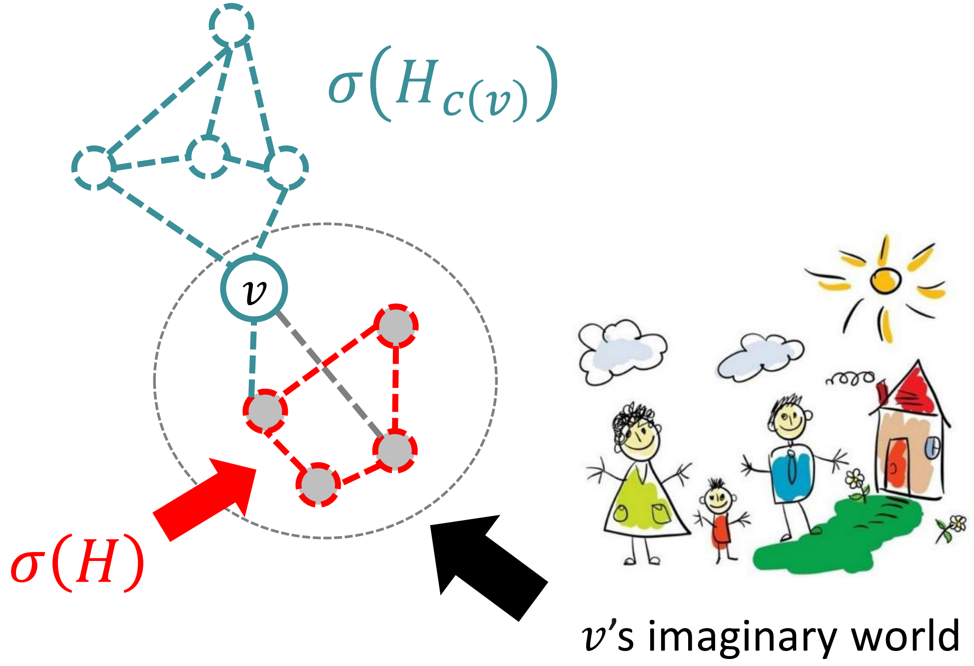


$\sigma(H)$ takes up space

v's imaginary world

Correctness Proof

- ✓ <u>Lemma 1:</u> $\sigma(H)$ is inside the imaginary world of some vertex v
- ✓ <u>Lemma 2:</u> $\sigma\left(H_{c(v)}\right) \not\subseteq v$'s imaginary world
- <u>Lemma 3:</u> $v \in \sigma\left(H_{c(v)}\right)$



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✓<u>Lemma 1:</u> $\sigma(H)$ is inside the imaginary world of some vertex v

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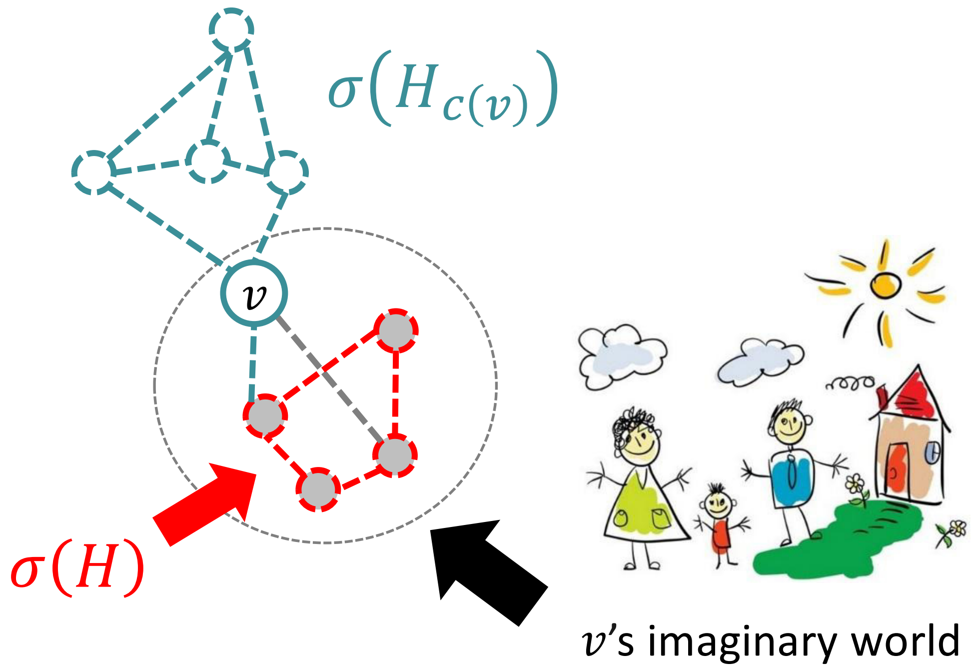
- <u>Lemma 3:</u> $v \in \sigma(H_{c(v)})$
- <u>Proof:</u>
  - $\sigma(H) \subseteq v$'s imaginary world



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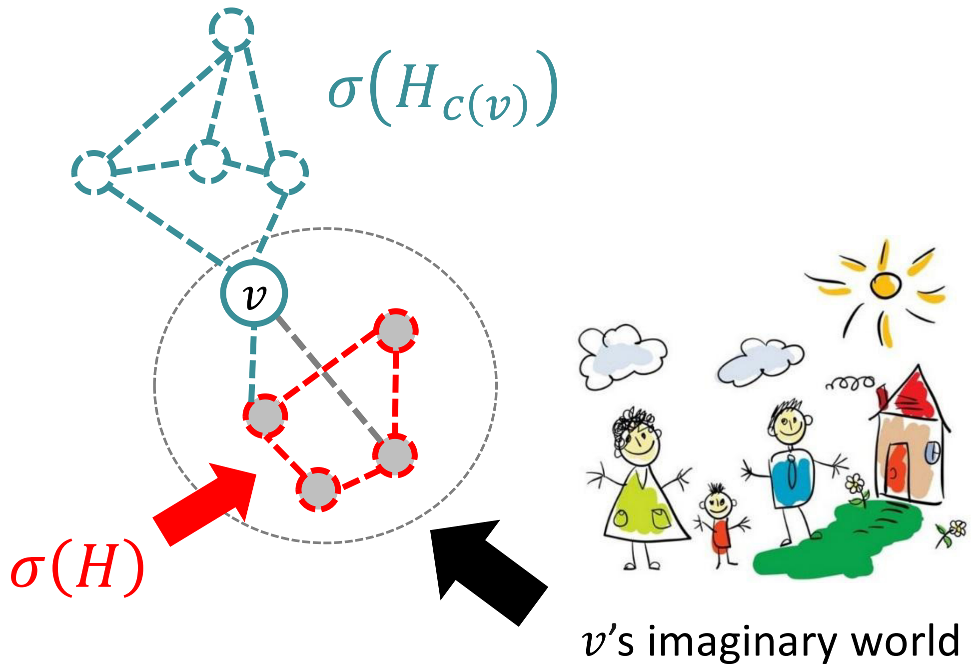
- <u>Proof:</u>
  - $\sigma(H) \subseteq v$'s imaginary world
  - $\sigma\left(H_{c(v)}\right)$ spills out



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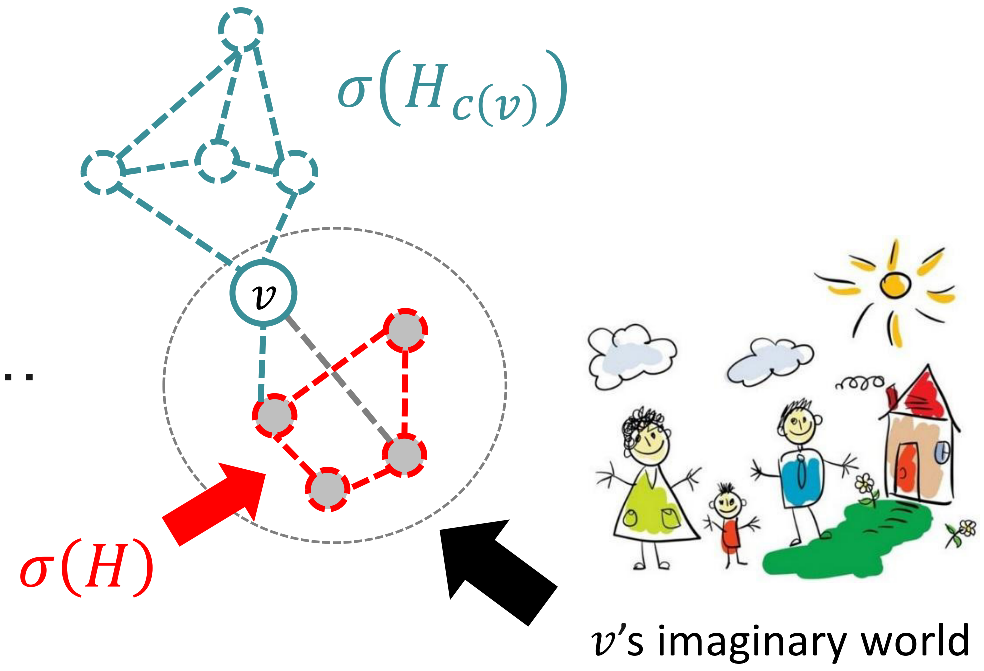
- <u>Proof:</u>
  - $\sigma(H) \subseteq v$'s imaginary world
  - $\sigma\left(H_{c(v)}\right)$ spills out
  - Can't get to $\sigma\left(H_{c(v)}\right)$ except through v…



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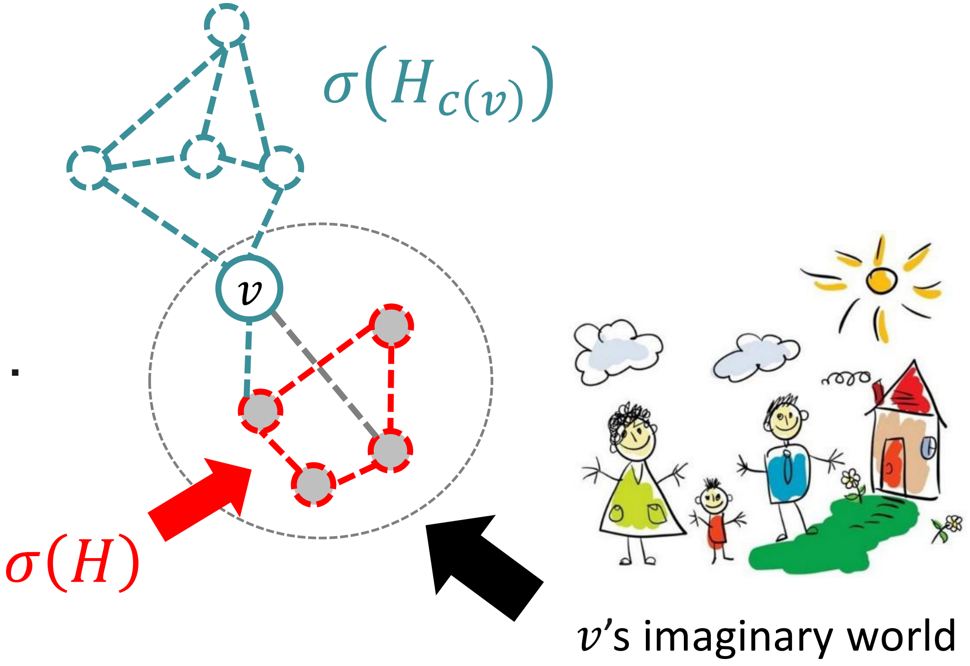
- <u>Lemma 3:</u> $v \in \sigma(H_{c(v)})$
- <u>Proof:</u>
  - $\sigma(H) \subseteq v$'s imaginary world
  - $\sigma(H_{c(v)})$ spills out
  - Can't get to $\sigma(H_{c(v)})$ except through v…
  - So $v \in \sigma(H_{c(v)})$



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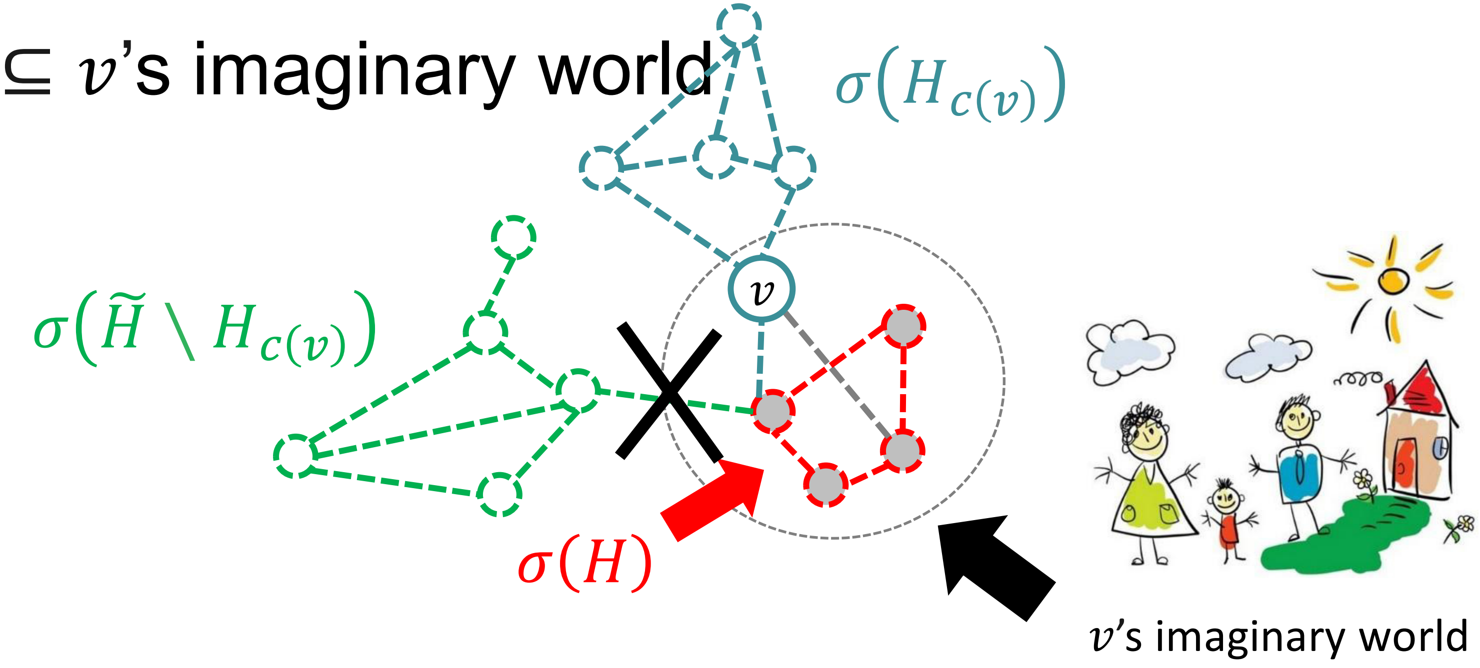
- <u>Lemma 4:</u> $\sigma(\widetilde{H} \setminus H_{c(v)}) \subseteq v$'s imaginary world



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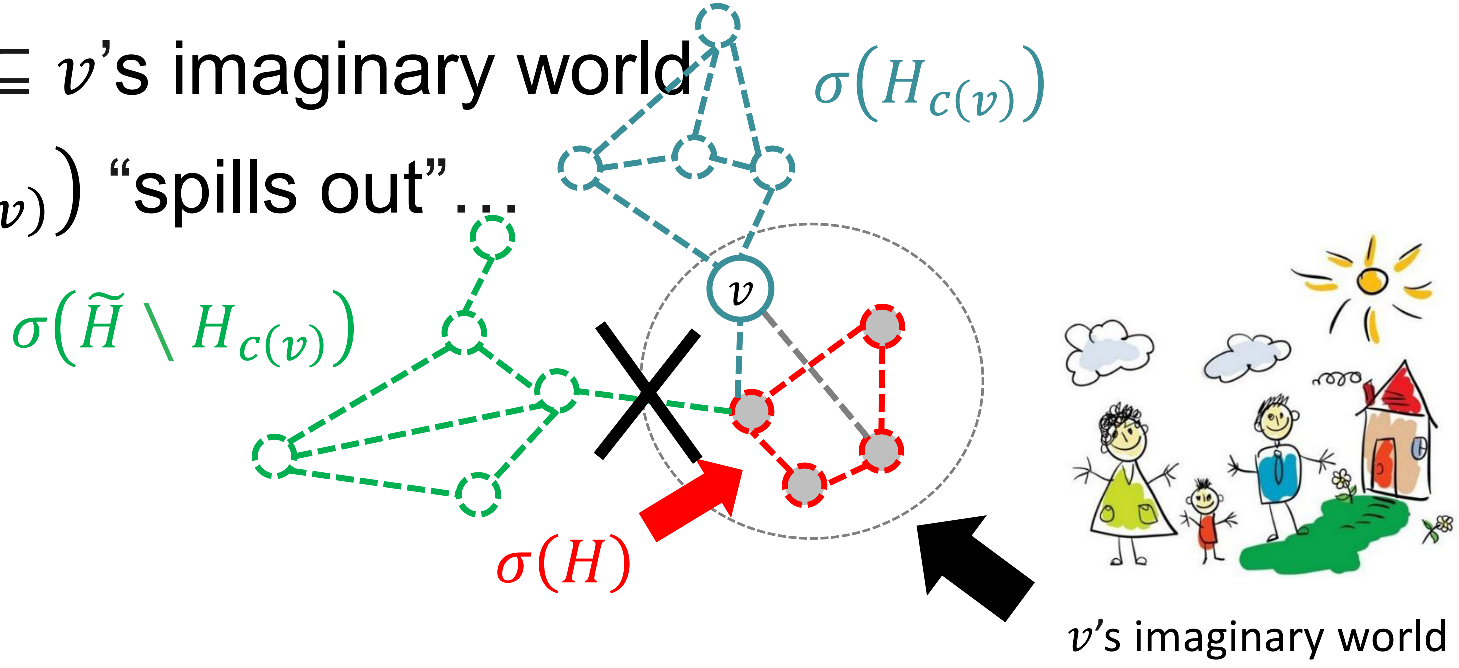
- <u>Proof:</u> suppose $\sigma(\widetilde{H} \setminus H_{c(v)})$ "spills out"…



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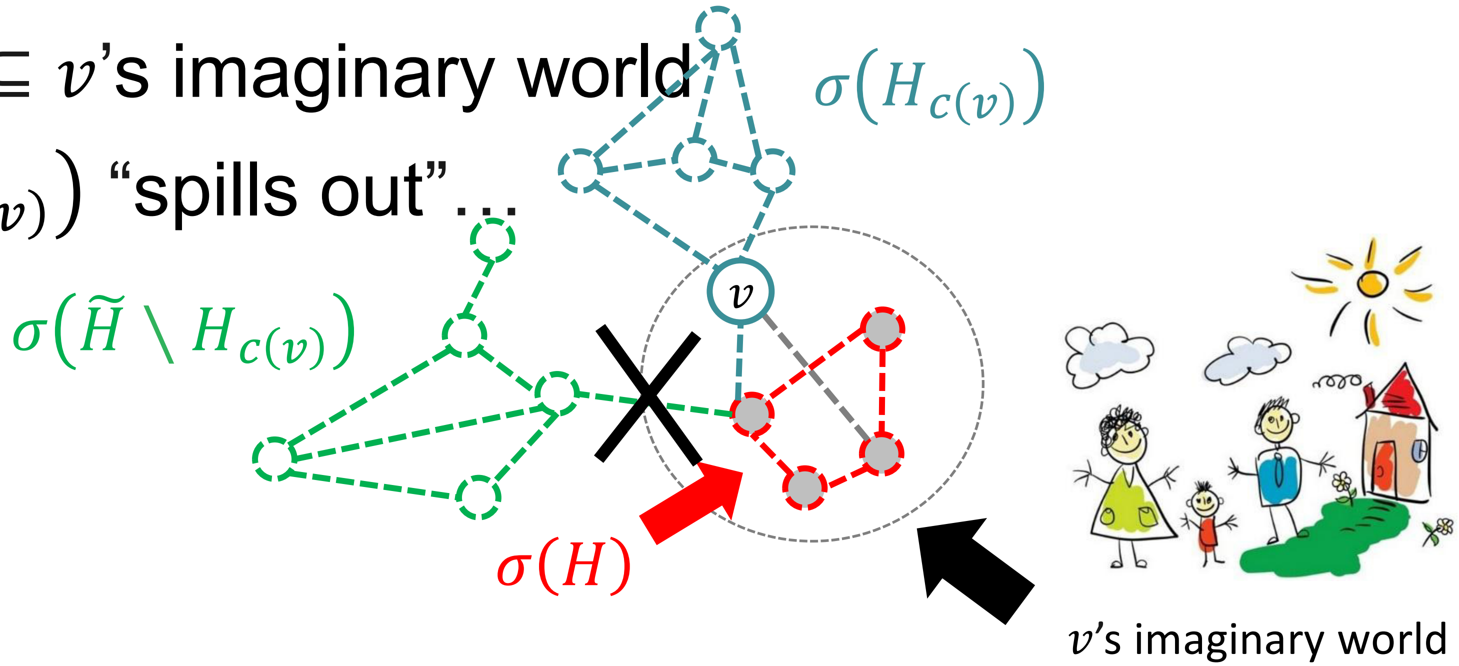

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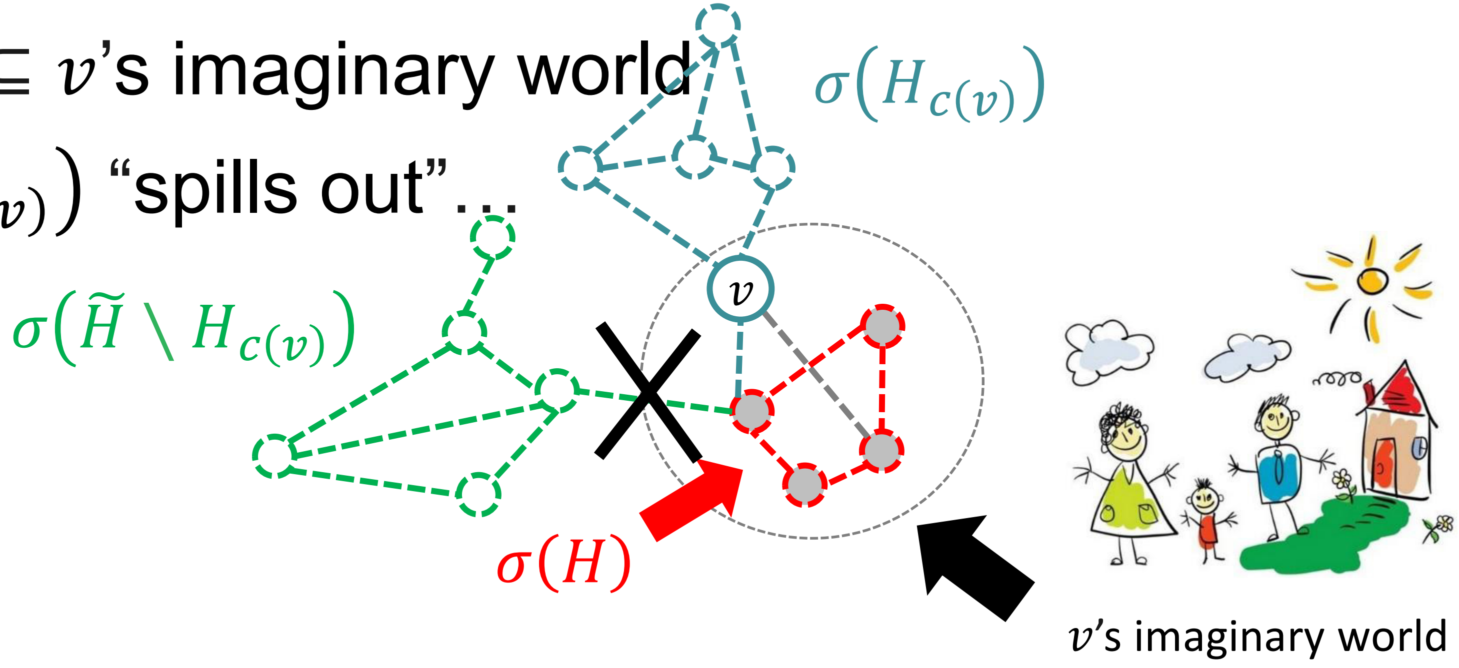
- <u>Proof:</u> suppose $\sigma\left(\widetilde{H} \setminus H_{c(v)}\right)$ "spills out"…
  - But v is the only exit!
  - And v is already taken…
    since $v \in \sigma\left(H_{c(v)}\right)$



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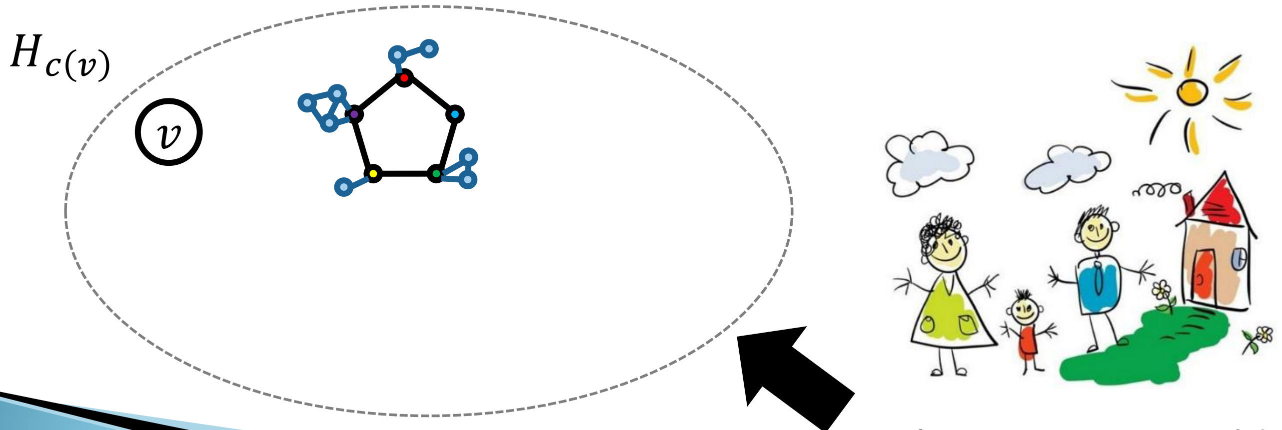
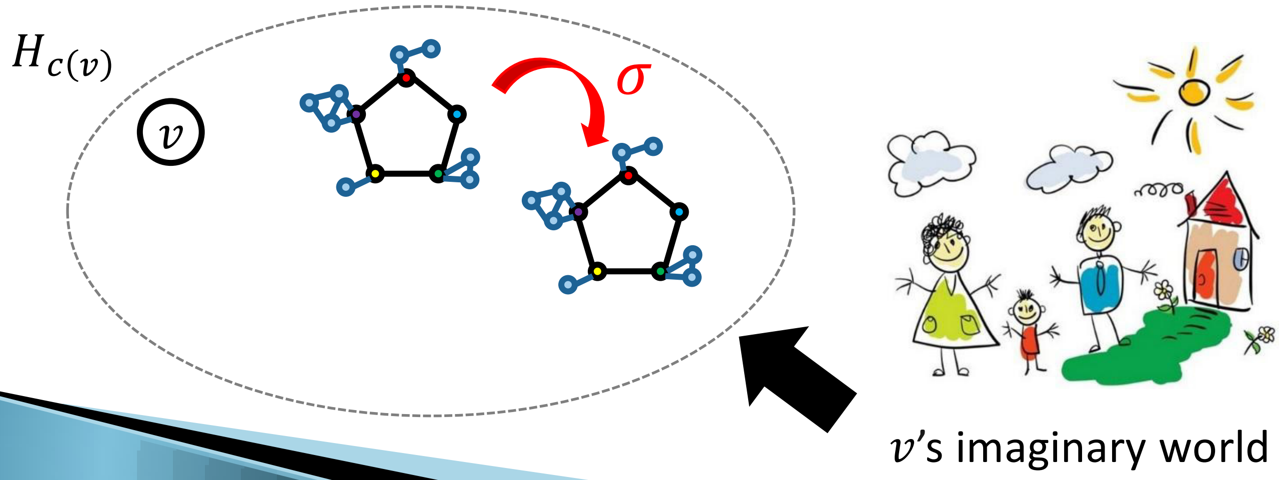


v's imaginary world

- So… $H_{c(v)}$ contains a copy of $\widetilde{H} \setminus H_{c(v)}$???
- What does σ do to this copy???

Correctness Proof

- <u>Claim:</u> σ maps the copy of $\widetilde{H} \setminus H_{c(v)}$ inside $H_{c(v)}$ to a new copy of $\widetilde{H} \setminus H_{c(v)}$ inside v's imaginary world
  - Can't re-use the old copy
  - Can't spill out



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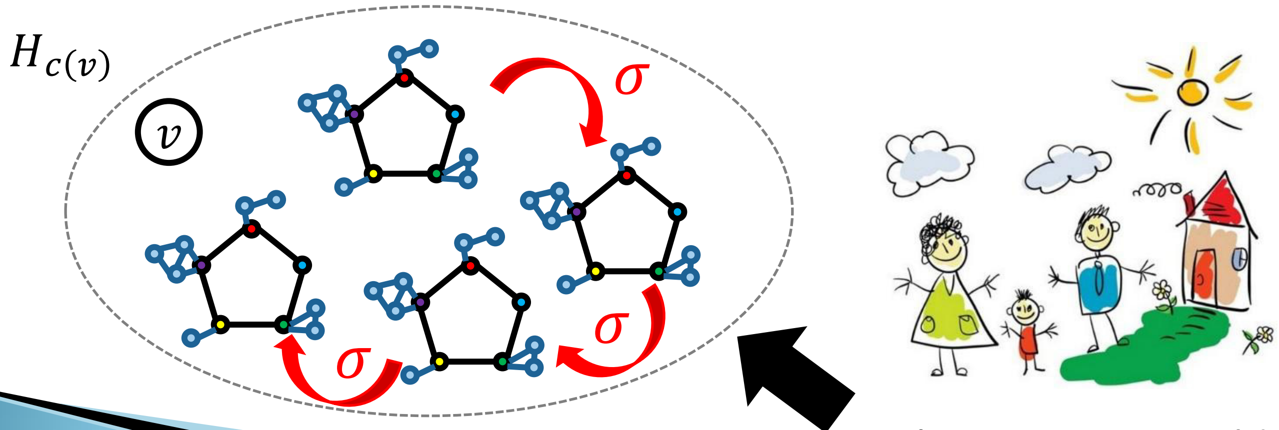
- By induction…



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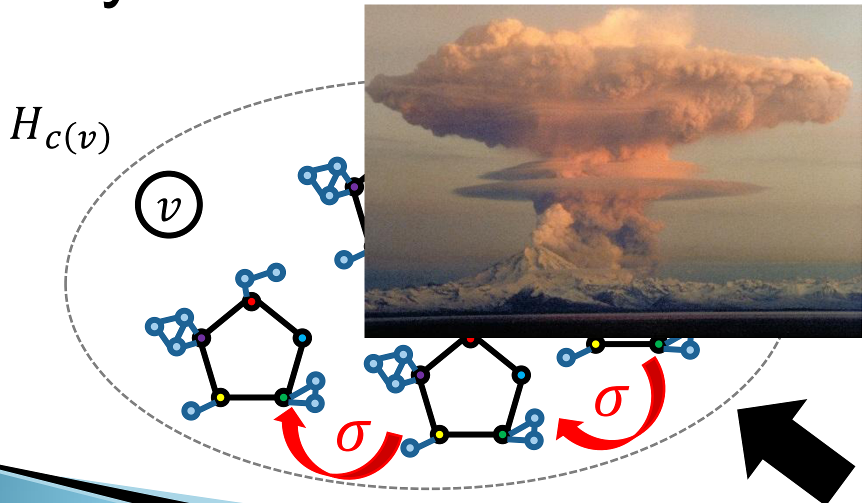

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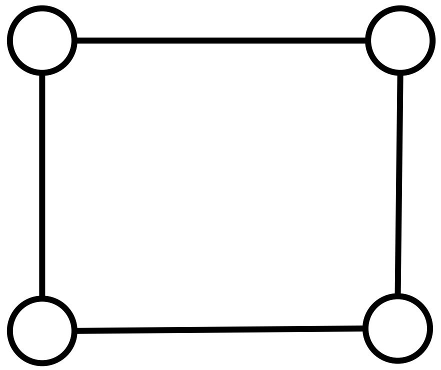
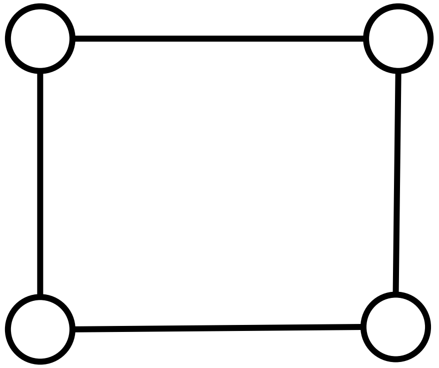
Reduction #2: Edge Replacement - Reminder

- Take a 4-cycle graph



Reduction #2: Edge Replacement - Reminder

- Take a 4-cycle graph
- Replace each edge by an arbitrary graph such that the whole graph is a 2-connected graph



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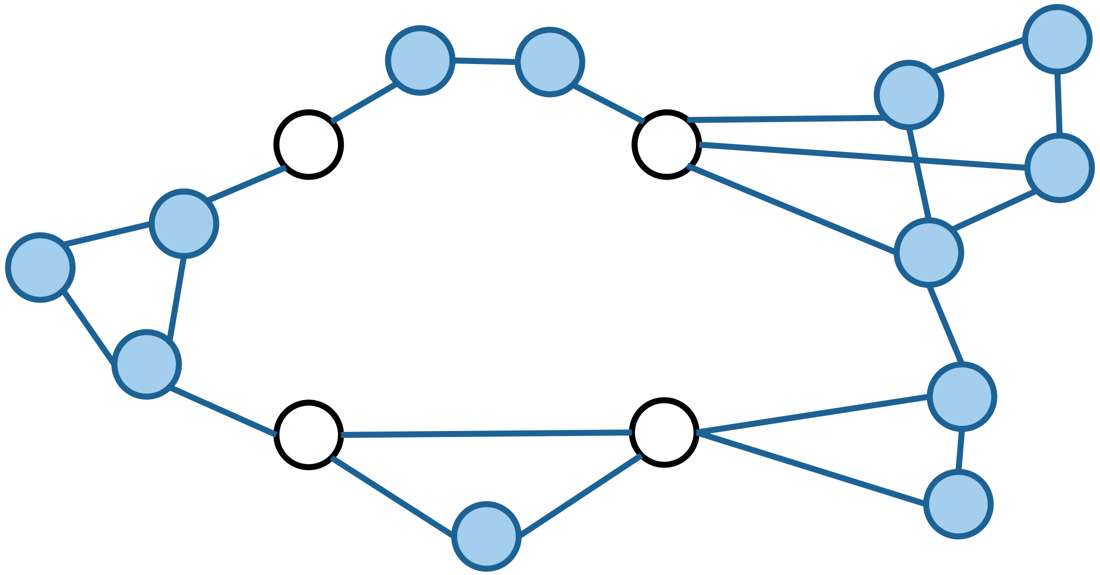
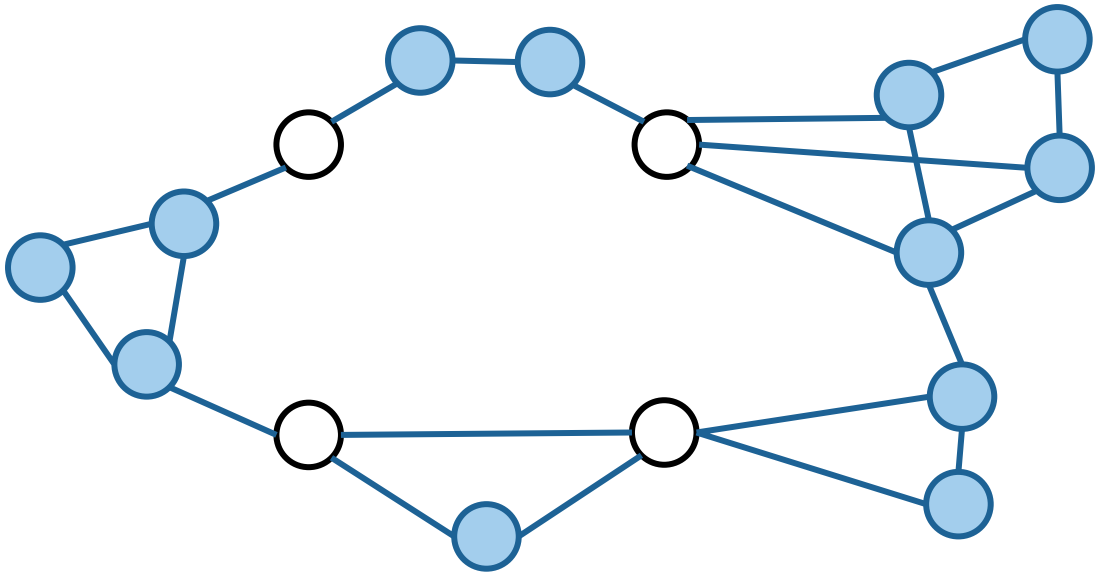


Reduction #2: Edge Replacement - Reminder

- Take a 4-cycle graph
- Replace each edge by an arbitrary graph such that the whole graph is a 2-connected graph
- We call this class of graphs class B

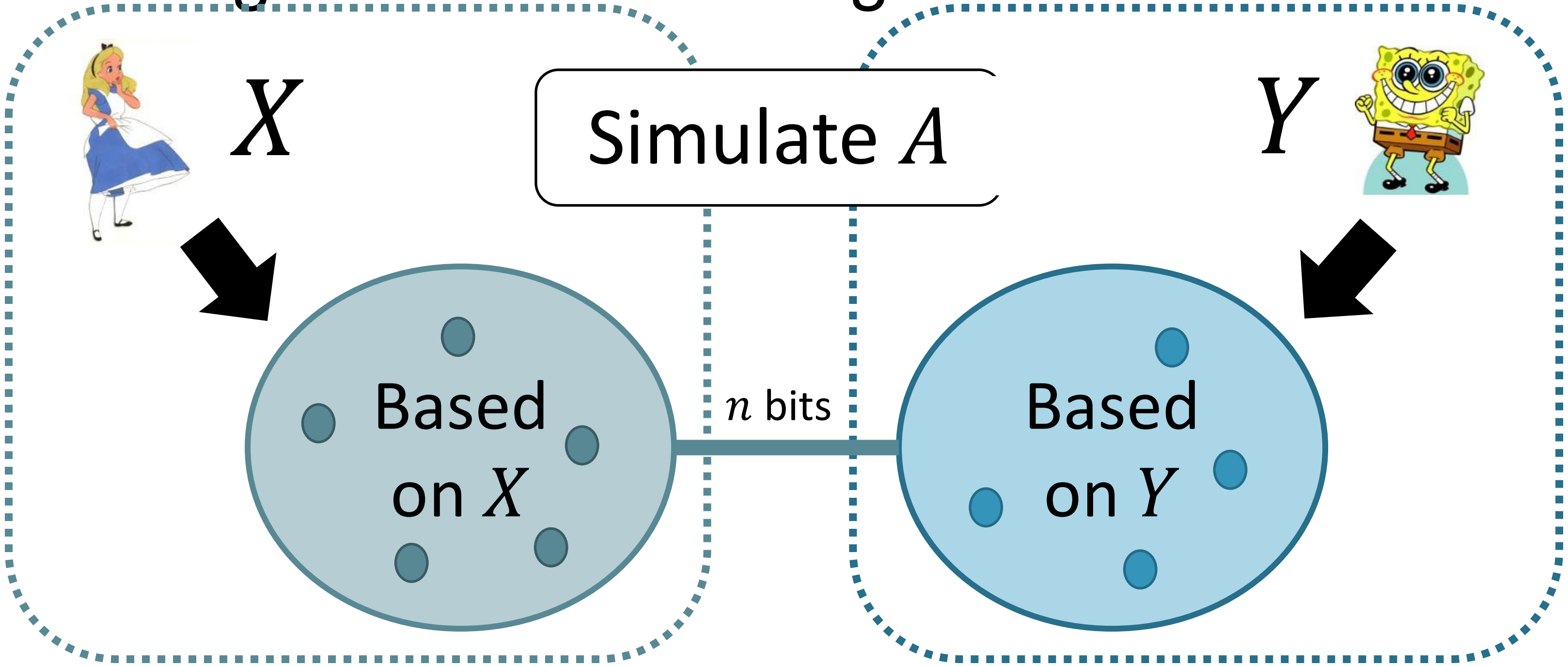


2-Party Communication Complexity

Applying Two-Player Communication Complexity Lower Bounds

Textbook reduction [Kushilevitz-Nisan]:

Given algorithm A for solving task T…



Solution for T ⇨ answer for DISJOINTNESS